# Truncated Wigner approximation for the Lee-Huang-Yang corrections to the Bogoliubov ground state


**K. L. Ng**[1], M. B. Kruk[1] and P. Deuar[1]

[1]IFPAN Quantum Noise Group, Institute of Physics, Polish Academy of Sciences, Warszawa, Poland


## Introduction

We demonstrate that the beyond-mean-field Lee-Huang-Yang (LHY) corrections [1] to the Bogoliubov ground-state [2] can alternatively be described by implementing an effective interaction strength $g_0$ and an effective condensate density $n_0$ in the truncated Wigner approximation. This treatment gives identical expected initial energy and density condition for the mean-field and the Wigner field dynamics. Numerical simulations of the Bose-Einstein condensate (BEC) dynamics in several one-dimensional systems are carried out accordingly using the beyond-mean-field description and the truncated Wigner approximation. For the example of applying a periodic trapping potential to an initially homogeneous BEC, the truncated Wigner method shows mostly qualitative agreement with the beyond-mean-field result and shows that the interference pattern developed in the beyond-mean-field is unphysical. This allows replacing the beyond-mean-field LHY correlations with the truncated Wigner appproximation, and leads to the determination of the quantum correlations for systems which the LHY corrections are non-negligible.

## Effective interaction strength and density

The standard Hamiltonian for a weakly interacting dilute gas has the form $\hat{H} = \int dx \left[-\hat{\Psi}^\dagger(x)\frac{\hbar^2\nabla^2}{2m}\hat{\Psi}(x) + V(x)\hat{\Psi}^\dagger(x)\hat{\Psi}(x) + \frac{g}{2}\hat{\Psi}^{\dagger 2}(x)\hat{\Psi}^2(x)\right]$. To take into account quantum fluctuations, the Bose field operator is expanded as $\hat{\Psi}(x) = \phi_0(x) + \delta\hat{\phi}(x)$. The Hamiltonian is diagonalized by the Bogoliubov transformation, $\delta\hat{\phi}(x) = \frac{1}{\sqrt{V}}\sum_k \left[u_k\hat{b}_k e^{ik\cdot x} - v_k\hat{b}_k^\dagger e^{-ik\cdot x}\right]$, where $u_k = \frac{\varepsilon_k + E_k}{2\sqrt{\varepsilon_k E_k}}$ and $v_k = \frac{\varepsilon_k - E_k}{2\sqrt{\varepsilon_k E_k}}$ for homogeneous gas with $E_k = \hbar^2k^2/2m$ and $\varepsilon_k = \sqrt{E_k(E_k + 2g_0 n)}$.

The higher-order energy contributions of the quasiparticle lead to the LHY corrections [3]. Note that the LHY correction is dimension-dependent, for 1D [4]:

$$E_{\text{LHY}}(g,n) = -\int dx \frac{2gn\sqrt{mgn}}{3\pi\hbar}.$$

For beyond-mean-field description, the total energy of a homogeneous BEC system consists of the mean-field interaction, $E_{\text{int,MF}} = \int dx \frac{gn^2}{2}$, and the LHY correction, i.e.

$$E_{\text{BMF}}(g,n) = E_{\text{int,MF}}(g,n) + E_{\text{LHY}}(g,n).$$

The truncated Wigner method describes quantum system by introducing vacuum noise to the initial state. For Bogoliubov ground state, the phonon modes are represented as complex Gaussian random variables $\hat{b}_k \sim \beta_k$ and $\hat{b}_k^\dagger \sim \beta_k^*$ where $\langle|\beta_k|^2\rangle = \frac{1}{2}$, i.e. [5,6]

$$\Psi_W(x) = \phi_0(x) + \frac{1}{\sqrt{L}}\sum_k \left(u_k\beta_k e^{ikx} - v_k\beta_k^* e^{-ikx}\right).$$

For Bogoliubov ground state in Wigner representaion, the expected total energy consists of the kinetic term $E_{\text{kin,W}} = \frac{L}{4\pi}\frac{\hbar^2}{m}\int dk|v_k|^2k^2$ and the interaction term $E_{\text{int,W}} = \int dx \frac{g_0 G^{(2)}}{2}$:

$$E_{\text{W}}(g_0,n) = E_{\text{kin,W}}(g_0,n) + E_{\text{int,W}}(g_0,n),$$

where $G^{(2)} = n^2 - \frac{n}{V} + 4n\delta n + 2\text{Re}(\bar{m})\sqrt{n(n-\frac{1}{V})} + |\bar{m}|^2 + 2(\delta n)^2$ is the pair correlation, $\bar{m} = \frac{1}{V}\frac{L}{2\pi}\int dk u_k^* v_k$ is the anamolous pair density and $\delta n = \frac{1}{V}\frac{L}{2\pi}\int dk|v_k|^2$ is the density depletion. The effective interaction strength $g_0$ is determined when:

$$E_{\text{W}}(g_0,n) = E_{\text{BMF}}(g,n).$$

The effective condensate density $n_0$ is used to retain the mean-field density:

$$n_0 = n - \delta n(g_0,n) = \phi_0^2.$$

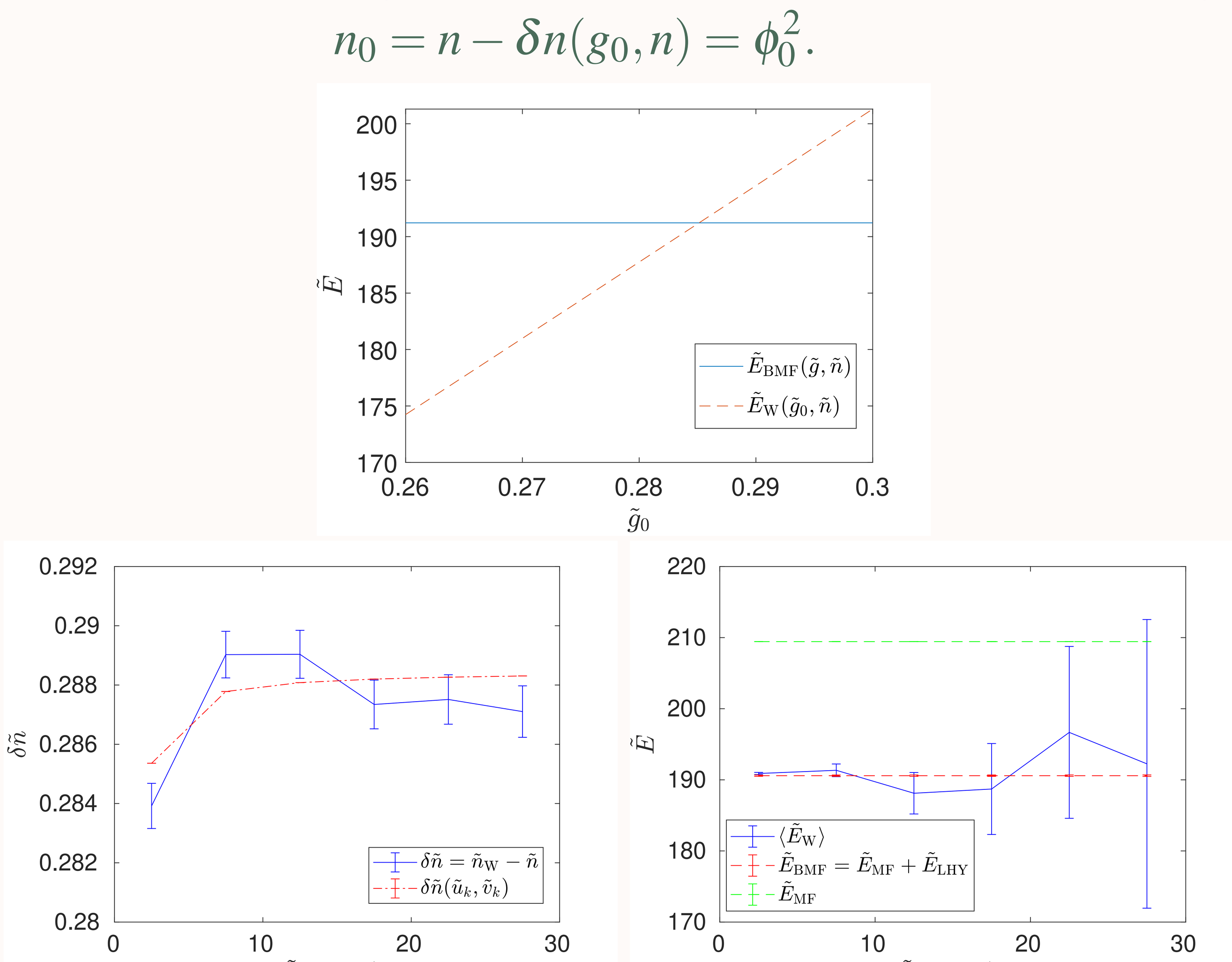


Figure 1: Top: matching the expected total energy from the Wigner method with the beyond-mean-field energy to determine the effective interaction strength $\widetilde{g}_0$. In this example $\widetilde{g} = 0.3$, $\widetilde{g}_0 \approx 0.285$ (in dimensionless form, $\widetilde{x} = \frac{x}{\xi_h}$ in the unit of healing length $\xi_h = \frac{\hbar}{\sqrt{2mng}}$). Bottom left: the density depletion predicted by the Wigner method matches to the expected depletion determined from the Bogoliubov coefficients $\widetilde{u}_k, \widetilde{v}_k$. Bottom right: Wigner simulation retains the LHY correction, here $\widetilde{E}_{\text{LHY}} \approx -18.86$.

## Dynamics of the BEC

Taking into account quantum fluctuations beyond the Gross-Pitaevskii equation, the time evolution of the one-dimensional mean-field (in dimensionless form $\widetilde{\psi}_{MF}$) follows the extended Gross-Pitaevskii equation (EGPE) which includes the LHY correction:

$$\frac{\partial\widetilde{\psi}_{MF}}{\partial\widetilde{t}} = -i\left[-\widetilde{\nabla}^2 + \widetilde{V} + \widetilde{g}|\widetilde{\psi}_{MF}|^2 - \frac{1}{\sqrt{2}\pi}\widetilde{g}^{\frac{3}{2}}|\widetilde{\psi}_{MF}|\right]\widetilde{\psi}_{MF}.$$

Using the effective interaction strength $\widetilde{g}_0$ and the effective condensate density $\widetilde{n}_0$, the Wigner field ($\widetilde{\psi}_W$) follows the standard GPE without the LHY correction:

$$\frac{\partial\widetilde{\psi}_W}{\partial\widetilde{t}} = -i\left[-\widetilde{\nabla}^2 + \widetilde{V} + \widetilde{g}_0|\widetilde{\psi}_W|^2\right]\widetilde{\psi}_W.$$

## Comparison between EGPE and Wigner method

The Wigner field dynamics with the effective $\widetilde{g}_0$ and $\widetilde{n}_0$ is compared to the mean-field dynamics. The LHY correction in mean-field is replaced by the quantum fluctuations introduced in the Wigner field. The two effective parameters retain the energy and the density of the bose gas system as the LHY correction employed.

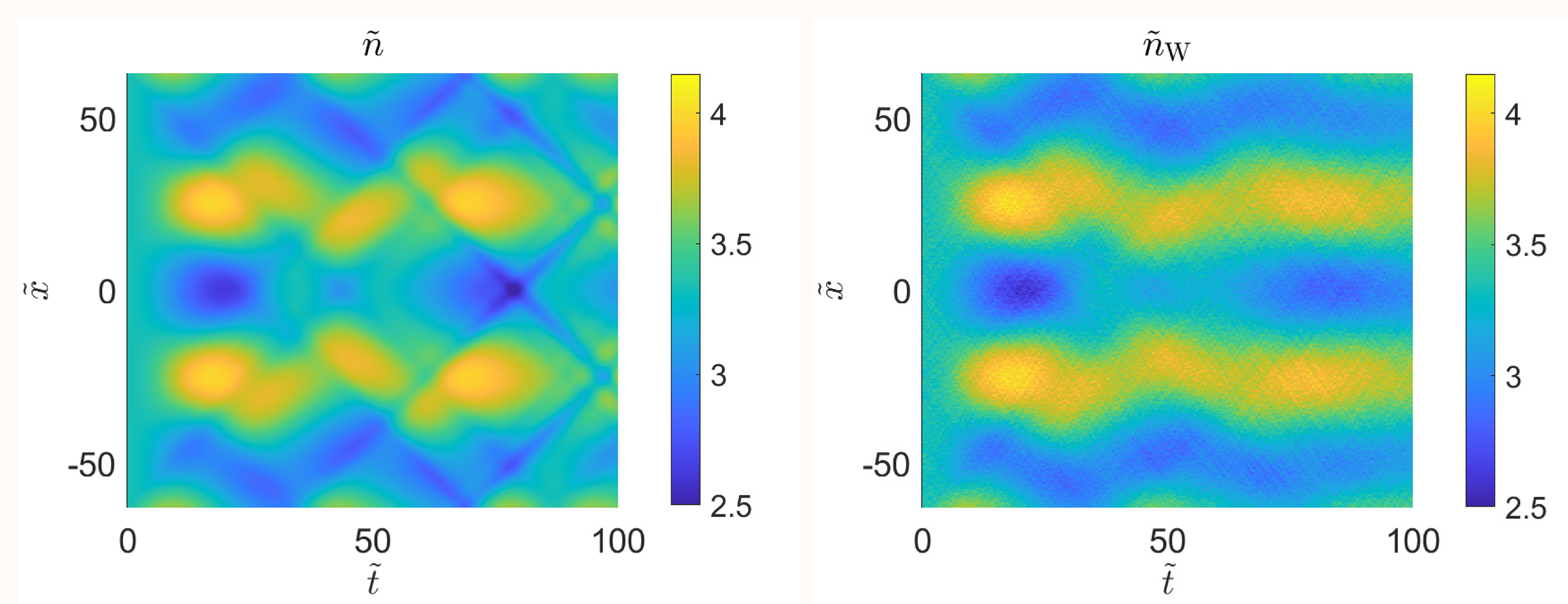


Figure 2: Simulations of the dynamics of an initially homogeneous bose gas, a periodic trapping potential $\widetilde{V} = \frac{\widetilde{g}\widetilde{n}}{10}\cos(\frac{\widetilde{x}}{8})$ is applied at $\widetilde{t} > 0$. Left: EGPE prediction, sharp interference pattern developed at $\widetilde{t} \gtrsim 50$. Right: the truncated Wigner approximation shows that the interference pattern is unphysical.

## $k$-space correlations

A harmonic potential $\widetilde{V} = 4\widetilde{g}\widetilde{n}(\frac{2\widetilde{x}}{L})^2$ is applied to an initially homogeneous bose gas. For mean-field and beyond-mean-field which the LHY correction is included, pair correlation $g^{(2)} = 1$. Using the truncated Wigner approximation, pair correlations can be determined. Here $k-$space pair correlations are calculated.

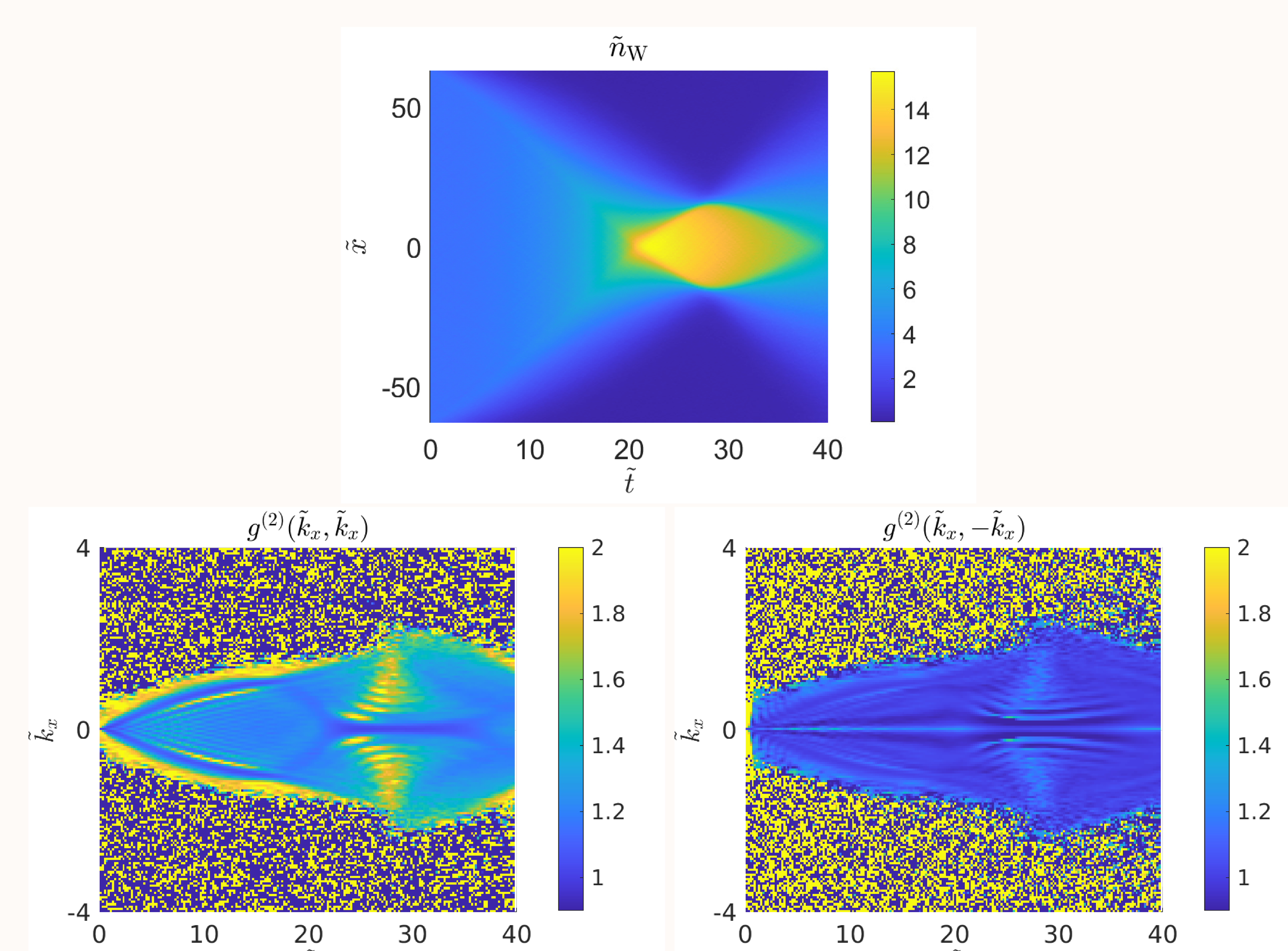


Figure 3: Simulations of the time evolution of an initially homogeneous bose gas with a harmonic potential applied at $\widetilde{t} > 0$. Top: gas density profile simulated by truncated Wigner method. Bottom left: pair correlation $g^{(2)}(\widetilde{k}, \widetilde{k})$. Bottom right: pair correlation $g^{(2)}(\widetilde{k}, -\widetilde{k})$. A strong correlation is developed after maximum compression $25 \lesssim \widetilde{t} \lesssim 30$.

## References


1. K. Huang et al., Phys. Rev. 105, 767 (1957).
2. N. Bogolubov, J. Phys. (USSR) 11, 23 (1947).
3. C. Weiss et al., Z. Naturforsch. 59a, 1 – 13 (2004).
4. D. Rakshit et al., New J. Phys. 21, 073027 (2019).
5. A. D. Martin et al., Phys. Rev. Lett. 104, 194102 (2010).
6. K. L. Ng et al., J. Phys. A: Math. Theor. 52, 035302 (2018).