Spontaneous emission in a Fabry–Perot cavity: frequency modulation and collective effects

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Abstract

The spontaneous emission of radiation from two two-level atoms in a Fabry–Perot cavity is considered. The transition frequency of the atoms is assumed to be modulated. It is shown that the frequency modulation leads to radical change of the spontaneous emission dynamics, including the decoherence properties of the entangled atomic states. Two ingredients are necessary in order to achieve this. Firstly, the modulation should be able to periodically detune the atoms from the resonance with the cavity. Secondly, the dynamics of spontaneous emission is efficiently modified if a resonant condition between the modulation frequency and modulation depth is fulfilled. Thus, frequency modulation can be used to dynamically control the collective effects in spontaneous emission in cavities.

1. Introduction

Spontaneous emission is beyond any doubt one of the most important electromagnetic processes which appear in nature. Even though the fundamental breakthrough works by Einstein [1] and Dirac [2] provided the explanation of the process either statistically, or on the basis of a fundamental theory, the details of the process remain of considerable interest even today. Until recently, spontaneous emission used to be one of the major obstacles preventing efficient control of atoms, and is still considered to be an obstacle to be overcome en route to build a working quantum computer [3]. On the other hand, spontaneous emission is also one of the most remarkable causes of decoherence spoiling quantum effects associated with the superposition principle. From this point of view, spontaneous emission very efficiently destroys entangled states of the atoms as well as any Schrödinger-cat states that may exist in atomic systems. Thus, the process of spontaneous emission is also of critical importance for understanding quantum mechanics since no interpretation of quantum theory can neglect the effect of decoherence, and one of the interpretations, championed among others by Zurek [4], heavily relies on it.

The problem of spontaneous emission in cavities and resonators is also of particular interest, see, e.g., [5, 15], because cavities allow for remarkable control of the interactions between the atoms and the electromagnetic field. Besides the more or less standard problem of spontaneous emission from a single non-moving atom in a cavity, the effects of atomic motion have been studied [16], and the collective effects have also been investigated in [17].

It is well known that frequency modulation of either the atomic transition (which can be achieved with the help of an external strongly off-resonant field, see, e.g.,[18]), or the field mode(s) themselves (e.g. via the adiabatic periodical motion of the cavity walls) can lead to a number of interesting effects [19, 23]. In particular, in a 1999 paper Agarwal [24] has shown that the decay and decoherence of a single atomic system in a heat bath can be controlled using the frequency modulation. What we consider to be a valuable continuation of Agarwal’s paper is the study of collective effects in atomic systems with frequency modulation coupled to the reservoir which consists
of the electromagnetic modes of a high-$Q$ cavity. This work can also be viewed as a further development of our earlier studies of the decoherence of multi-mode Schrödinger-cat states [25, 26]. Using the perturbative method of multiple time scales we obtain an explicit asymptotic solution of the time-dependent Schrödinger equation. As a by-product of the multiple-scales analysis a non-Hermitian Hamiltonian describing interaction between the atoms and explicit decay rates are also obtained.

The main body of this paper is organized as follows. In section 2, we describe our model and recall its solution for the case of vanishing phase modulation. Section 3 contains the multiple-scales asymptotic analysis of the problem for the most interesting set of values of the key parameters. In section 4, our numerical results are presented, and section 5 contains some final remarks.

2. The model

Let us consider two identical two-level non-moving atoms located inside a linear cavity of Fabry–Perot type. The cavity is assumed to be quasi-one-dimensional and bounded by a perfect mirror at one side and by a partially transparent input mirror at the other side. It is assumed that the transmittivity of the input mirror is sufficiently weak. For frequencies close to that of the atomic transition frequency the electric field operator can be written as

$$E(x) = \mathcal{E}(x) \cos(qx) \int_0^\infty dk \left( g(k)c_k + g^*(k)c_k^\dagger \right),$$

where

$$\mathcal{E} = \sqrt{\frac{h\omega_c}{Ve_0}},$$

and $V$ is the volume of the cavity, $\omega_c$ is the resonance frequency and $q = \omega_c/c$. In the following we shall use such a system of units that $\hbar = 1$. What is more restrictive, we assume the Lorentzian profile of the spectral response function of the cavity, that is

$$g(k) = \sqrt{\frac{\Gamma}{\pi}} \frac{1}{\Gamma + i(k - \omega_c)}.$$  

The natural canonical commutation relations between $c_k$ and $c_k^\dagger$ must hold:

$$[c(k), c^\dagger(k')] = \delta(k - k').$$

It is assumed that both atoms in the cavity have modulated frequencies with the modulation depth $f$ and frequency $\Omega$, that is, the Hamiltonian of the atoms without coupling with the field reads

$$H_A = (\omega_0 - f \sin(\Omega t))(\sigma_{ee}^{(1)} + \sigma_{ee}^{(2)}),$$

where $\omega_0$ is the transition frequency without modulation, superscripts (1) and (2) refer to the first and the second atom (considered to be distinguishable), and $\sigma_{ee} = |e\rangle\langle e|$, with $|e\rangle$ denoting the excited atomic state, and the notation $\sigma_{eg}, \sigma_{ee}, \sigma_{gg}$ is obvious.

The free-field part of the Hamiltonian is given by

$$H_F = \int_0^\infty k dk c_k^\dagger c_k,$$

while the interaction Hamiltonian in the rotating wave approximation reads

$$H_{IF} = g_0 \sum_{j=1}^2 \cos(qx_j) \int_0^\infty dk (g(k)c_k^\dagger \sigma_{ge}^{(j)} + g^*(k)\sigma_{eg}^{(j)}c_k),$$

where $x_j$ is the location of the $j$th atom in the cavity. The total Hamiltonian $H = H_A + H_F + H_{IF}$ can be divided into two parts, $H = H_0 + H_I$ so that

$$H_0 = \omega_0 \left( 1 - \frac{f}{\omega_0} \sin(\Omega t) \right) \left( \sigma_{ee}^{(1)} + \sigma_{ee}^{(2)} + \int_0^\infty dk c_k^\dagger c_k \right).$$

The interaction Hamiltonian in the interaction representation with respect to $H_0$ is then given by

$$H_I = \int_0^\infty dk \left( (k - \omega_0)c_k^\dagger c_k + \right.$$

$$\left. + g_0 \sum_{j=1}^2 \cos(qx_j) (g(k)c_k^\dagger \sigma_{ge}^{(j)} e^{-i\omega \cos \Omega t} + \text{h.c.}) \right)$$

where $a = f/\Omega$.

There are at least three different physical mechanisms which may lead to an effective modulation of atomic frequency. Firstly, the atoms may be subject to an external strongly off-resonant laser field. In that case the modulation amplitude $f$ is equal to the ratio

$$\Omega_m^2/(2\omega_0^2),$$

where $\Omega_m$ is the Rabi frequency of the off-resonant field and $\omega_0$ is the ‘bare’ atomic frequency, see [27], section 4.2. The frequency $\omega_0$ which appears in (3) already contains the static part of the Stark shift caused by the modulating field:

$$\omega_0 = \omega_0 + \Omega_m^2/\omega_0.$$

Secondly, one of the cavity mirrors may oscillate sinusoidally under the influence of a piezoelectric element associated with it. Let the mirror at rest occupy the position $x = -L/2$, while the instantaneous location of the second mirror be $L/2 + A \sin(\Omega t)$. In this case, it is not the atomic frequency, but rather the central cavity frequency which is modulated. The atomic Hamiltonian is simply

$$H_A = \omega_0 (\sigma_{ee}^{(1)} + \sigma_{ee}^{(2)}).$$

Building the Hamiltonian for the cavity field in the case of the cavity with oscillating mirrors is not a trivial task. However, if the amplitude of the mirror oscillations $A$ is very small when compared to the size of the cavity at rest $L$ (e.g. $\delta = A/L \sim 10^{-4}$, which is a realistic value) and the frequency of the mirror oscillations $\Omega_m$ much smaller than the central cavity frequency $\omega_0$ (so that $\Omega/\omega_0$ is again of the order of $10^{-6}$), the Hamiltonian for the cavity field can be written approximately as

$$H_F = \int dk (k - \delta_0 \sin(\Omega t)) c_k^\dagger c_k,$$

provided that $\Gamma \ll \omega_0$. A simple unitary transformation brings again the interaction Hamiltonian (6). However, the interpretation of the quantity $\alpha$ is now different. It is not just
the ratio between the modulation amplitude and the modulation frequency, but contains four quantities characterizing the cavity:

\[ a = \frac{A}{\sqrt{L/\Omega}}. \]

On the other hand, \( \Omega \) retains its meaning as the modulation frequency.

Thirdly, the cavity may contain inside it a dielectric with a time-dependent dielectric constant. Let the dielectric occupy the place between \( x = -L/2 \) and \( x = -L/2 + d \). Let us also assume that, for \( -L/2 \leq x \leq -L/2 + d \), the dielectric constant varies in time according to

\[ n^2(t) = n_0^2 + \delta \sin(\Omega t), \]

where \( \delta \) is again very small, being of the order of \( 10^{-4} \). In this case the field Hamiltonian can again be written in the form of (7) (see, e.g., [28]) except that \( \omega_c \) is modified due to the presence of the dielectric (without taking account of the modulation) in the cavity.

In this work we consider a two-atom system such that only one of the atoms is initially excited. Thus, the interaction-modulation) in the cavity.

Let us now introduce a pseudo-mode picture. The wave vector function can be represented as

\[ \Psi_f = \alpha_{1f}|\psi_1(0)\rangle + \alpha_{2f}|\psi_2(0)\rangle + \int dk g(k) \beta_1|\psi_1(0)\rangle + \beta_2|\psi_2(0)\rangle, \]

where subscripts 1, 2 indicate to which atom a particular state refers, the state \( |0\rangle_f \) is the vacuum state of the cavity field, and \( |1_k\rangle_f \) denotes the state of the field with one photon of the frequency \( k \) present in the cavity.

The Schrödinger equation for the wave vector \( \Psi_f \) gives the following differential equations for the time-dependent amplitudes \( \alpha_j \) and \( \beta_k \):

\[ i\dot{\alpha}_1 = g_0 d_1 \int dk g(k)^2 \beta_k e^{i \omega_c \cos \Omega t}, \]

\[ i\dot{\alpha}_2 = g_0 d_2 \int dk g(k)^2 \beta_k e^{i \omega_c \cos \Omega t}, \]

\[ i\dot{\beta}_k = (k - \omega_0) \beta_k + g_0 [d_1 e^{i \omega_c \cos \Omega t} \alpha_1 + d_2 e^{-i \omega_c \cos \Omega t} \alpha_2], \]

where \( d_1 = \cos(\alpha t) \).

Let us now introduce a pseudo-mode \( B \) defined by

\[ B(t) = \int_0^\infty dk g(k)^2 \beta_k(t). \]

Then one can easily verify that the dynamics of \( \alpha_1 \) and \( \alpha_2 \) given by equations (9)–(11) is the same as that governed by the following system of differential equations:

\[ \dot{\alpha}_1 = -ig_0 d_1 e^{i \omega_c \cos \Omega t} B, \]

\[ \dot{\alpha}_2 = -ig_0 d_2 e^{i \omega_c \cos \Omega t} B, \]

\[ B = -i(\Delta - i\Gamma) B - ig_0 [d_1 \alpha_1 + d_2 \alpha_2] e^{-i \omega_c \cos \Omega t}, \]

where \( \Delta = \omega_c - \omega_0 \). Because of the initial conditions for \( \beta_k \), \( \beta_k(0) = 0 \), we also have \( B(0) = 0 \). The only approximation used to derive equations (13)–(15) consists in extending the region of integration over \( k \) to \( -\infty \). This is justified because the spectral profile \( g(k) \) is well centred near \( \omega_c \), and this quantity, together with \( \omega_0 \), is the largest characteristic frequency of the system.

Let us first consider the case of vanishing modulation (\( \alpha = 0 \)). In this case equations (13)–(15) can easily be solved with the help of the Laplace transformation \( \mathcal{L} \). For the Laplace-transformed amplitudes \( \tilde{\alpha}_1(s) \) and \( \tilde{\alpha}_2(s) \):

\[ \tilde{\alpha}_1(s) = \frac{s(s + i(\Delta - i\Gamma^2)) + g_0^2 d_1^2}{s(s + i(\Delta - i\Gamma^2)) + g_0^2 (d_1^2 + d_2^2)} \alpha_0(0) - g_0^2 d_1 d_2 \alpha_0(0) \]

\[ \tilde{\alpha}_2(s) = \frac{s(s + i(\Delta - i\Gamma^2)) + g_0^2 d_2^2}{s(s + i(\Delta - i\Gamma^2)) + g_0^2 (d_1^2 + d_2^2)} \alpha_0(0) - g_0^2 d_1 d_2 \alpha_0(0). \]

The roots of the denominator in the above equations give the frequencies and decay rates of the amplitudes \( \alpha_1 \), \( \alpha_2 \):

\[ s_0 = 0, \]

\[ s_{\pm} = \frac{1}{2} \left[ -i \Delta - \Gamma \pm i \sqrt{(\Delta - i\Gamma)^2 + 4g_0^2 (d_1^2 + d_2^2)} \right]. \]

It is clear that the resulting behaviour of the amplitudes \( \alpha_j \) in time strongly depends on the relation between \( g_0 \) and \( \Gamma \). In the generic case we have to do with damped vacuum Rabi oscillations. The dependence of \( \alpha_j(t) \) on the initial conditions is non-trivial, as will be illustrated in several figures in section 4. Our model in which the atoms are coupled with just one cavity mode allows for the stationary, non-decaying states if \( d_1 \alpha_1 + d_2 \alpha_2 = 0 \) at \( t = 0 \).

Let us investigate the asymptotic behaviour of \( \alpha_1 \) for large times. Because the dynamics of the pseudo-mode \( B \) is damped, we can expect that, asymptotically, \( B \) shall simply follow \( \alpha_1 \) and \( \alpha_2 \), that is:

\[ B \approx -\frac{g_0}{\Delta - i\Gamma} (d_1 \alpha_1 + d_2 \alpha_2). \]

As a result, for very long times the amplitudes \( \alpha_j \) exhibit damped oscillations with the damping rate equal to

\[ g_0^2 (d_1^2 + d_2^2), \]

and, as \( t \to \infty \), they approach the values:

\[ \alpha_1 = \frac{d_2 \alpha_2(0) - d_1 \alpha(0)}{d_1^2 + d_2^2}, \]

\[ \alpha_2 = \frac{-d_1 \alpha_1(0) + d_2 \alpha_2(0)}{d_1^2 + d_2^2}. \]

Those simple analytical results agree with the numerical ones presented in section 4.

3. Results of multiple-scales analysis in the presence of phase modulation

Let us now consider the system (13)–(15) for \( \alpha \neq 0 \).

To analyse the above system of differential equations we introduce a dimensionless time \( \tau = \Omega t \). It is assumed that \( g_0, \)
\( \Delta \) and \( \Gamma / \Omega \) are all much smaller than \( \Omega \). We write
\[
g_0 = \epsilon \gamma_0, \quad \Delta / \Omega = \epsilon \delta, \quad \Gamma / \Omega = \epsilon \mu.
\]

Then equations (13)–(15) take the form:
\[
\begin{align*}
\dot{a}_1 &= -i \epsilon \gamma_0 d_1 e^{i \omega \cos \tau} B \\
\dot{a}_2 &= -i \epsilon \gamma_0 d_2 e^{i \omega \cos \tau} B \\
\dot{B} &= -i (\delta - i \mu) B - i \epsilon \gamma_0 [d_1 a_1 + d_2 a_2] e^{-i \omega \cos \Omega \tau}.
\end{align*}
\]

The key formula to perform the calculation is the expansion of the exponent \( \exp(\pm i \omega \cos \tau) \), \( \tau = \Omega t \), in terms of the Bessel functions:
\[
\exp(\pm i \omega \cos \tau) = J_0(a) + \sum_{n=1}^{\infty} (\pm i)^n J_n(a) \cos(n \tau).
\]

This means that, apart from the fast varying terms containing multiples of \( \Omega \), the atom–field coupling constant is modified (or ‘dressed’) by the external modulating field to become \( g_0 J_0(a) \) instead of \( g_0 \). Let us observe that if
\[
J_0(a) = 0,
\]
the right-hand sides of equations (13)–(15) contain only the fast varying terms plus the first term in equation (15). This is, naturally, related to the fact that the interaction Hamiltonian can then be written as
\[
H_I = \int_0^\infty \! dk \left[ (k - \omega_0) c_k^\dagger c_k + g_0 \sum_{j=1}^2 \cos(qx_j) \left( g(k) c_k^\dagger \sigma_{ij}^{(j)} \sum_{n=1}^\infty (-i)^n J_n(a) + \text{h.c.} \right) \right],
\]
which means that, apart from the diagonal contribution from the field modes, the interaction Hamiltonian contains only the fast varying terms. This means that the dynamics of the atomic subsystem will be effectively ‘turned off’ in the time scales of the order of at least \( 1/\Omega \). However, if \( g_0 \ll \Omega \), we can expect that the evolution of internal states of the atoms is stabilized for a much longer time, of the order of \( \Omega / g_0^2 \). Both analytical and numerical calculations confirm this last prediction.

To analyse the problem mathematically, let us introduce several time scales according to
\[
T_0 = \tau, \quad T_\mu = \epsilon^2 \tau,
\]
and expand the amplitudes \( \alpha_j \) and \( B \) in terms of \( \epsilon \):
\[
\begin{align*}
\alpha_j &= \alpha_j^{(0)} + \epsilon \alpha_j^{(1)} + \epsilon^2 \alpha_j^{(2)} + \cdots, \\
B &= B^{(0)} + \epsilon B^{(1)} + \epsilon^2 B^{(2)} + \cdots.
\end{align*}
\]

It seems that the algebra involved in the multiple-scales expansion here is quite tedious and not particularly enlightening (calculations are fairly similar to those sketched at the end of section 3 in [27]). However, let us mention here that one of the reasons the multiple scales are useful is that they make it possible to avoid the so-called secular terms in approximate solutions which are (unphysically) polynomial in time. In our case, the condition that such terms do not arise does not bring any special relations in the first- and second order of the perturbation expansion in \( \epsilon \). However, in the third order the secular terms do not arise if only the following equations for \( \alpha_j^{(n)} \) are fulfilled:
\[
\frac{\partial \alpha_j^{(n)}(T_3)}{\partial T_3} = -i \gamma_0 \delta - i \mu \cos(qx_j) \cdot 2 \sum_{n=1}^\infty \frac{J_n^2(a)}{n^2} \times \left\{ \cos(qx_j) \alpha_j^{(0)}(T_3) + \cos(qx_j) \alpha_j^{(2)}(T_3) \right\},
\]
\[
\frac{\partial \alpha_j^{(2)}(T_3)}{\partial T_3} = -i \gamma_0 \delta - i \mu \cos(qx_j) \cdot 2 \sum_{n=1}^\infty \frac{J_n^2(a)}{n^2} \times \left\{ \cos(qx_j) \alpha_j^{(0)}(T_3) + \cos(qx_j) \alpha_j^{(2)}(T_3) \right\}.
\]

Let us now attempt to find an effective Hamiltonian which generates the dynamics of \( \alpha_j^{(n)} \), \( n = 1, 2 \), in the time \( T_3 \). That is, we define the effective interaction-picture, zeroth-order wavefunction as
\[
\Psi^{(0)} = \alpha_j^{(0)}(1) |g_2 \rangle + \alpha_j^{(2)}(1) |e_2 \rangle.
\]

Let us now define the effective interaction Hamiltonian between the atoms as such an operator \( h_{\text{eff}} \) that the Schrödinger equation for \( \Psi^{(0)} \)
\[
\frac{1}{\Delta} \frac{\partial \Psi^{(0)}}{\partial \tau} = h_{\text{eff}} \Psi^{(0)}
\]
gives precisely equations (26), (27) for the amplitudes \( \alpha_j^{(n)} \).

That effective Hamiltonian, which generates the dynamics in \( T_3 \), can be written in the form
\[
h_{\text{eff}} = \sum_{i,j,k,l} h_{ijkl}(x_1, x_2) \sigma_i^{(1)} \sigma_j^{(2)} \sigma_k^{(1)} \sigma_l^{(2)},
\]
and it has the following four matrix elements:
\[
\begin{align*}
h_{\text{eggg}} &= 2 \gamma_0^2 \delta - i \mu \cos(qx_1) \frac{\sum_{n=1}^\infty J_n^2(a)}{n^2}, \\
h_{\text{eggx}} &= 2 \gamma_0 \delta - i \mu \cos(qx_2) \frac{\sum_{n=1}^\infty J_n^2(a)}{n^2}, \\
h_{\text{eggx}} &= 2 \gamma_0 \delta - i \mu \cos(qx_1) \cos(qx_2) \frac{\sum_{n=1}^\infty J_n^2(a)}{n^2}.
\end{align*}
\]

The above Hamiltonian must be necessarily non-Hermitian because of the damping. Let us stress here that it has been derived, and not postulated. One can immediately observe that the effective interatomic interactions very heavily depend on the environment, that is, on its spatial, spectral and also temporal properties. Firstly, the effective interactions depend not only on the difference \( x_1 - x_2 \), but also on the sum \( x_1 + x_2 \), which is quite intuitive in the case of cavity problems. Secondly, the spectral characteristics enter via the parameters \( \delta \) and \( \mu \). Thirdly, it is the parameters which characterize the temporal properties of the environment (including the external field or motion of cavity walls) that are responsible for the fact that \( h_{\text{eff}} \) is of the third order of smallness with respect to \( \epsilon \).
The final results for the zeroth-order amplitudes \( a_j^{(0)} \) are given by
\[
\alpha_j^{(0)}(t) = \frac{1}{d_1^2 + d_2^2} \left[ \left( d_1^2 + d_2^2 e^{2i\omega_t} \right) a_j^{(0)}(0) - d_1 d_2 (1 - e^{-2i\omega_t}) a_j^{(0)}(0) \right],
\]
\[
\alpha_j^{(0)}(0) = \frac{1}{d_1^2 + d_2^2} \left[ -d_1 d_2 (1 - e^{-2i\omega_t}) a_j^{(0)}(0) + (d_1^2 + d_2^2 e^{-2i\omega_t}) a_j^{(0)}(0) \right],
\]
where the expression for the quantity \( S \) is rather intuitive:
\[
S = \frac{g_0^2}{\Omega^2} \left( \Delta - i\Gamma \right) \left( d_1^2 + d_2^2 \right) \sum_{n=1}^{\infty} \frac{J^2(a_n)}{n^2}.
\]

Although the limit \( t \to \infty \) cannot be taken literally in the above expressions, let us still note that such a limit would agree with that given in the last equations of the previous section. From (34) it is clear that both the damping rate and radiative frequency shift are influenced by the phase modulation in such a way that they appear only in the second order with respect to the small parameter \( g_0/\Omega \), only if the resonant condition (24) is fulfilled. Thus, the spontaneous decay of the atoms in general, and the decoherence of entangled states in particular, will be strongly suppressed if we manage to appropriately modulate the atomic frequency. More precisely, let us choose \( a = 2.40483 \), being the first zero of the \( J_0 \) function. We have here in mind microwave cavities interacting with atoms in highly excited states with \( \omega_0 \approx \omega_0 \approx 10^9 – 10^{10} \text{ Hz} \). In the good cavity limit we can have \( g_0 \approx 10^4 – 10^5 \text{ Hz} \) and \( \Gamma \) can be as small as \( 10^4 – 10^5 \text{ Hz} \). We need to take into account, however, that the frequency modulation (be it achieved by an application of an external non-resonant field or by the motion of a mirror) necessarily leads to some heating of the system [29]; thus we should consider an intermediate-coupling regime with \( g_0 / \Gamma \). This does not, however, change our prediction as long as we manage to achieve the modulation frequency \( \Omega \) and the modulation depth \( f \) of the order of \( 10^9 – 10^5 \text{ Hz} \), and keep \( \Gamma \) well below this value.

Needless to say, the modulation with the frequency substantially larger than \( \Gamma \) causes the atom to be effectively sometimes in, and sometimes out of resonance with the cavity field. However, as we can already guess from the above analytical formulae, such a ‘switching’, though necessary, is on its own insufficient to effectively prevent the decay. One should also have the condition (24), because the effective coupling constant is proportional to \( J_0(a) \).

The above results, on one hand, confirm those already obtained by Agarwal [24]. On the other hand, we have obtained the explicit formulae for the decay and frequency shift, and more importantly, have been able to quantitatively characterize the effective coupling between the atoms in cavity with phase modulation under the resonance condition (24).

4. Wigner function: evolution of the phase distribution

The Wigner function is a very convenient tool to describe the quantum states of simple systems. Here we would like to employ it to investigate the dynamics of the phase relations between the two atoms in the cavity.

Unlike in the case of harmonic oscillator, there are several definitions of the Wigner function for the spin or two-level atom systems. Here we work with the Wigner function defined via the Schwinger boson representation of the angular momentum operators [30, 31].

Following Schwinger, let us represent the angular momentum operators in terms of pairs of boson operators:
\[
J_x = (1/2)(a^\dagger b + b a),
\]
\[
J_y = (i/2)(a^\dagger b - b^\dagger a),
\]
\[
J_z = (1/2)(a^\dagger a - b^\dagger b).
\]

The operators \( a, a^\dagger, b, b^\dagger \) satisfy canonical commutation relations:
\[
[a, a^\dagger] = [b, b^\dagger] = 1,
\]
and all other commutators vanish. The operators \( J_x, J_y, J_z \) satisfy the commutation relations of angular momentum. The eigenstates for arbitrary spin \( j \) are constructed as
\[
|j, m \rangle = \frac{(a^\dagger)^m (b^\dagger)^{j-m}}{\sqrt{j + m + 1}! j!} |0, 0, 0 \rangle,
\]
where \( |0, 0, 0 \rangle \) is a rotationally invariant state with zero angular momentum; here it plays the role of a vacuum from which the states with non-zero angular momentum are created with the help of the creation operators \( a^\dagger, b^\dagger \). In particular, for \( j = 1/2 \) the state \( |1/2, 1/2 \rangle \) can be identified with the excited state of a two-level atom, and the state \( |1/2, -1/2 \rangle \) with the ground state of that atom. With regard to the commutation relations, the Pauli operators \( \sigma_x, \sigma_y, \sigma_z \) can be expressed in terms of \( a, a^\dagger, b, b^\dagger \) in the same way as \( J_x, J_y, J_z \), except that the factor \( 1/2 \) should be dropped.

The Wigner function can be conveniently constructed via the characteristic function which, in the case of \( N \) atoms, is defined as (cf [26]):
\[
C(\eta, \eta^*, \zeta, \zeta^*) = \text{Tr} \left[ \rho_A \exp \left( \sum_{i=1}^{N} (\eta_i a_i^\dagger + \zeta_i b_i^\dagger - \eta_i^* a_i - \zeta_i^* b_i) \right) \right],
\]
where \( \rho_A \) is the density matrix of the atomic subsystem, and \( i = 1, 2, \ldots, N \). In our case of two spontaneously emitting atoms the above definition brings the following expression for the (time-dependent) characteristic function:
\[
C(\eta, \eta^*, \zeta, \zeta^*) = \exp \left[ -\frac{1}{2} \sum_{i=1}^{2} (|\eta_i|^2 + |\zeta_i|^2) \right] \times \left[ 1 - (|\eta_1^2 + |\alpha_2|^2)(1 - |\zeta_1|^2)(1 - |\zeta_2|^2) \right] + |\alpha_1|^2(1 - |\eta_1|^2)(1 - |\zeta_1|^2) + |\alpha_2|^2(1 - |\eta_2|^2)(1 - |\zeta_2|^2) + \alpha_1 \alpha_2^* \eta_1 \zeta_1 \eta_2 \zeta_2 + \alpha_1^* \alpha_2 \eta_1^* \zeta_1^* \eta_2^* \zeta_2^*,
\]
where \( i = 1, 2 \). The corresponding Wigner function can obtained from the formula
\[
W(w_z, z) = \frac{1}{\pi^3} \int d^2\eta d^2\zeta d^2\zeta C(\eta, \zeta, \eta^*, \zeta^*) \times \exp \left[ \sum_{i=1}^{2} (w_i \eta_i + z_i \zeta_i - w_i^* \eta_i - z_i^* \zeta_i) \right].
\]
which gives

\[
W(w_i, w_i' z_i, z_i') = \frac{16}{\pi^3} \exp \left[ -2 \sum_{i=1}^{2} (|w_i|^2 + |z_i|^2) \right] 
\times \left[ (1 - (|\alpha_1|^2 + |\alpha_2|^2))(1 - 2(|z_1|^2 - 1)) + (1 - 2(|z_1|^2 - 1)) + |\alpha_2|^2(1 - 2(|w_1|^2 - 1)) + (1 - 2(|z_1|^2 - 1)) + |\alpha_1|^2(1 - 2(|w_1|^2 - 1)) + (1 - 2(|z_1|^2 - 1)) + 16|\alpha_1|^2|\alpha_2|^2|\zeta_1|^2|\zeta_2|^2 + 
\right]
\]

(38)

What we want to investigate with the help of the Wigner function is the phase relations between the two atoms. With this purpose in mind, we integrate the above Wigner function over \(|w_i|\) and \(|\zeta_i|\), \(i = 1, 2\), to obtain

\[
W_\eta(\psi, \chi) = \frac{1}{16\tau^2} \left[ 1 + \frac{1}{4} (|\alpha_1|^2 e^{i(\psi-\chi)} + |\alpha_2|^2 e^{i(\psi-\chi)}) \right],
\]

(39)

where \(\phi = \arg(\alpha_1) - \arg(\alpha_2), \psi = \psi_1 - \psi_2, \chi = \chi_1 - \chi_2\) and the phases \(\psi_i, \chi_i\) are defined via the relations \(w_i = |w_i| e^{i\psi_i}, z_i = |z_i| e^{i\chi_i}\). Thus, it is clear that the only phase-dependent part of \(W_\eta\) is contained in a cosine function which also contains the relative phase between the amplitudes \(\alpha_1, \alpha_2\). The argument of the cosine depends on the differences between the phases \(\psi_1 - \chi_1\) and \(\psi_2 - \chi_2\). Therefore, we cannot avoid the conclusion that \(\phi\) must be interpreted as the relative phase between two atoms. In the following section we provide some numerical results which illustrate how that relative phase depends on time for various depths of modulation.

5. Numerical results

To obtain the time evolution of the system for arbitrary (i.e. not necessarily small) values of \(\epsilon\) we have solved equations (20)–(22) numerically using a home-made algorithm based on the split-operator technique. The solutions have been obtained for several initial values of \(\alpha_1\) and \(\alpha_2\). The collection of figures given below illustrates the behaviour of the function of four interesting characteristics of the system. Needless to say, if we take into account the coupling of the atoms with all the modes of electromagnetic fields, both atoms would eventually decay to the tensor product of the ground states, so that the amplitudes \(\alpha_j, j = 1, 2\), must approach zero for large times. The time scale of that approach to zero is, however, much larger than any other time scales involved; hence it is not visible in the figures below.

In order to study the collective effects in modulated cavity, we have performed numerical calculations of the population \(P_1\) of the excited state of one of the atoms under the condition that only that atom is initially excited. In figure 1 we have compared the time dependence of \(P_1\) in two cases: (i) only one atom is present in the cavity (without and with modulation); (ii) two atoms pass through the cavity (modulated or not). Let us note here that, for point-like atoms, the first case is indistinguishable from the case of two atoms in a cavity under the condition that one of the atoms is in a nodal position of the cavity mode, hence the plot for \(d_2 = 0\). The number 2.40483 is an approximation of smallest zero of the zeroth-order Bessel function.

From figure 1 it is clear that the presence of the second atom can lead to the stabilization of \(P_1\). That stabilization is preserved by the modulation of the cavity, but the asymptotic value of \(P_1\) is reached after the time being two orders of magnitude longer. For both one- and two-atom dynamics with modulation, all the remnants of the damped Rabi oscillations disappear which is understandable because the system is governed by a completely different effective Hamiltonian. If we had included the coupling of the atoms with non-cavity modes, the stabilization would be converted to some plateau in the time dependence of \(P_1\) for the case of two atoms without modulation.

In figures 2 and 3 we have compared the results for population dynamics of the first atom under the condition that there are two atoms in the cavity, and for two different initial
conditions, \(a_1 = 1/\sqrt{2}\) and \(a_2 = \pm 1/\sqrt{2}\). This has been done for both non-modulated and modulated cavities, and for the nodal and anti-nodal positions of the second atom, so that four different curves appear in each figure. In the case \(a_1 = a_2 = 1/\sqrt{2}\) there is no stabilization in the population dynamics, regardless of the modulation, if \(d_1 = d_2\). In the opposite case \(d_2 = -1/\sqrt{2}\) (and \(d_1 = d_2\)) the dynamics in the two-atom case is actually frozen: since \(B(0) = 0\), the time evolution of \(\alpha_2\) cannot start. Therefore, two curves in figure 3 corresponding to \(d_2 = 1\) are the same whether the modulation is present or not.

From the point of view of the (de)coherence characteristics of the system, the quantity

\[
D(t) = 1 - \text{Tr}(\rho_A^2(t))
\]

(40)
is of some interest, because it can be thought of as a distance of the atomic state from a pure state (cf [32]). In the above equation, \(\rho_A\) denotes the reduced density matrix of the atom in the interaction representation. Taking into account the normalization condition for \(\Psi_1\):

\[
\langle \Psi_1 | \Psi_1 \rangle = |\alpha_1|^2 + |\alpha_2|^2 + \int dk |g(k)|^2 |\beta_k|^2 = 1,
\]

we find the following expression for \(D\):

\[
D(t) = 2(|\alpha_1(t)|^2 + |\alpha_2(t)|^2)(1 - (|\alpha_1(t)|^2 + |\alpha_2(t)|^2)).
\]

In figure 4 we have displayed the time dependence of the function \(D(\tau)\) with respect to time for \(\epsilon = 0.1\) (this is a fairly large value), \(\alpha_1(0) = 1, \alpha_2(0) = 0\) (thus the initial state is not entangled). It is clear that the condition (24) together with the fact that \(\Gamma\) is ten times smaller than \(\Omega\) leads to quite an effective increase of the time after which the distance of the atomic subsystem from the pure state is the largest. However, the most important information contained in figure 4 is probably that the stabilization of population of the excited states of two atoms happens when the state of the atomic subsystem is actually highly mixed. Resonant modulation leads to very considerable slowdown in both decay of populations and the approach to the maximally mixed state.

Figures 5 and 6 are provided to illustrate what happens if we change the initial conditions for the atomic amplitudes. The initial states chosen to plot those figures are both entangled. Figure 5 displays the time dependence of \(D(\tau)\) when the initial conditions for \(\alpha_1\) are \(\alpha_1(0) = 0 = \alpha_2(0)\) (this is a really dramatic change of the qualitative characteristic of the asymptotic state in the cavity even though the second atom is not coupled with the field in the cavity).

In figure 6 the time dependence of \(D(\tau)\) is modified with respect to what can be seen in figure 5 because the
In order to illustrate the dynamics of degradation of the relative phase between the atoms, we also display a parametric plot of \( R = \text{Re}(\alpha_1 \alpha_2^*) \) and \( I = \text{Im}(\alpha_1 \alpha_2^*) \), parametrized by the time \( \tau \).

One can observe in figure 10 that the resonant frequency modulation is very effective in keeping the relative phase close to its initial value. This, however, cannot be said about the non-resonant frequency modulation which creates quite large-amplitude oscillations of both \( R \) and \( I \). The above observations are valid for all pairs of initial values of \( \alpha_1, \alpha_2 \) such that \( \alpha_1 \alpha_2(0) + d_2 \alpha_2(0) \neq 0 \).

6. Final remarks

In this paper we have discussed the collective spontaneous emission from two two-level atoms in a Fabry–Perot cavity influenced by the periodic frequency modulation of the atomic transition frequencies. It has been shown that the frequency modulation may lead to considerable slowdown of the spontaneous emission and decoherence of entangled states.
states of the atoms. In particular, if a resonant condition which involves the modulation depth and the frequency of the (sinusoidal) modulation is fulfilled, the spontaneous emission becomes very slow. Effective decay rates and radiative frequency shifts have been obtained using a multiple-scales expansion. In addition, numerical examples showing the sensitivity of collective spontaneous emission on the phase of initial value of the wavefunction are provided. An effective Hamiltonian describing the coupling between spontaneously emitting atoms has been calculated. It turns out that the resonant frequency modulation also leads to a partial "turning off" of the interatomic interactions.

The phase relations between the two atoms have also been studied with the help of the atomic Wigner function obtained via the Schwinger bosonic representation of the atomic operators. It has been shown that those phase relations are kept near their initial values in the case of resonant modulation. It seems to us that the collective spontaneous decay and interatomic interactions in the Fabry–Perot cavities can be controlled to some extent by modulating the frequency of atomic transitions with the help of external fields, or, alternatively, by modulating the central frequency of the cavity.

In this work the atoms have been assumed to be point-like. It seems that the problem of collective dynamics of atoms with non-trivial center-of-mass wavefunctions might be of some interest. This, however, requires taking into account the atomic spin and corresponding statistics, because the atoms need not necessarily be distinguishable, as assumed here. We plan to study the problem of collective dynamics of Bose or Fermi atoms in a modulated cavity in a forthcoming publication.

References

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