Correspondence between dark solitons and the type II excitations of the Lieb-Liniger model

Tomasz Karpiuk,[1,2] Tomasz Sowiński,[3,4] Mariusz Gajda,[3,4] Kazimierz Rzążewski,[4] and Mirosław Brewczyk[1,4]
[1]*Wydział Fizyki, Uniwersytet w Białymstoku, ul. Lipowa 41, 15-424 Białystok, Poland*
[2]*Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543, Singapore*
[3]*Institute of Physics PAN, Al. Lotników 32/46, 02-668 Warsaw, Poland*
[4]*Center for Theoretical Physics PAN, Al. Lotników 32/46, 02-668 Warsaw, Poland*
(Received 24 February 2014; revised manuscript received 15 December 2014; published 21 January 2015)

A one-dimensional model of bosons with repulsive short-range interactions, solved analytically by Lieb and Liniger many years ago, predicts the existence of two branches of elementary excitations. One of them represents Bogoliubov phonons; the other, as suggested by some authors, might be related to dark solitons. On the other hand, it has been already demonstrated within the framework of the classical field approximation that a quasi-one-dimensional interacting Bose gas at equilibrium exhibits excitations which are phonons and dark solitons. By showing that for lower temperatures statistical distributions of dark solitons obtained within the classical field approximation are well represented by the distributions of quasiparticles of the second kind derived from fully quantum description, we demonstrate that type II excitations in the Lieb-Liniger model are, indeed, quantum solitons.

DOI: 10.1103/PhysRevA.91.013621 PACS number(s): 03.75.Hh, 03.75.Lm

I. INTRODUCTION

A relation between multiparticle quantum dynamics and its nonlinear classical field approximation is an intricate physical problem. A particularly interesting example is offered by a soluble model of particles moving on the circumference of a circle, interacting via contact potential known as the Lieb-Liniger model [1]. Its relation to a nonlinear single-particle dynamics has been recently investigated in [2,3].

In this paper we study thermal equilibrium properties of a one-dimensional uniform Bose gas in a weakly interacting regime. Our analysis is based on the exactly solvable model [1,4] and is also performed within the classical field approximation [5,6]. Our goal is to improve the understanding of the physical meaning of the type II branch of the elementary excitations inherently built in the Lieb-Liniger model [7]. Hence, we consider a system of N bosons on a circle interacting by repulsive delta-function potential. Surprisingly, such a system exhibits two families of excitations. As already identified by Lieb [7], the main branch belongs to phonons, which can be confirmed by comparing the corresponding energy spectrum to the one obtained within the Bogoliubov approach. This branch typically occurs in a three-dimensional case [8].

There have been attempts to associate the second branch of excitations to dark solitons [9–12] by analyzing the zero-temperature dispersion relation for solitary waves. There have been also beyond mean-field studies of the quantum nature of dark solitons (see [3], for example); however, none of them did explore the statistical distribution of excitations at nonzero temperatures. Here, we are going to establish the link between type II excitations and dark solitons by studying statistical distributions of excitations in a one-dimensional weakly interacting Bose gas at thermal equilibrium by using a multimode approach at nonzero temperatures.

II. ONE-DIMENSIONAL BOSE GAS AS A GAS OF QUASIPARTICLES

According to Landau and Lifshitz [13] the low-temperature equilibrium properties of the system can be discussed in terms of a gas of quasiparticles. Since for a one-dimensional Bose gas of atoms interacting via the contact forces the two families of excitations come out of the Lieb-Liniger model [7], a gas of two kinds of quasiparticles should be considered: phonons and type II excitations. These two kinds of quasiparticles coexist in a gas and interact with each other in an inelastic way. For low temperatures a good approximation is to assume that all properties of the system can be derived from the ideal quasiparticle gas model. In particular, we are interested in the type II excitations' distribution at a given temperature. For that it is convenient to write the state of the system in a number representation as $\{n_{p_1}^{(1)},n_{p_2}^{(2)}\}$. Here, $n_{p_1}^{(1)}$ and $n_{p_2}^{(2)}$ are the numbers of the Bogoliubov phonons and the type II excitations, respectively. Their momenta, $\{p_1,p_2\}$, according to Lieb [7], are quantized and equally spaced with the values being integer multiples of $2\pi/L$, where L is the size of the system. Moreover, the momenta of type II excitations are limited by $N\pi/L$. Total energy of the ideal gas of the quasiparticles is then given by

$$\mathcal{E}\{n_{p_1}^{(1)},n_{p_2}^{(2)}\} = \sum_{p_1} n_{p_1}^{(1)}\epsilon_1(p_1) + \sum_{p_2} n_{p_2}^{(2)}\epsilon_2(p_2), \qquad (1)$$

where excitations spectra $\epsilon_1(p_1)$ and $\epsilon_2(p_2)$ depend on the only relevant dimensionless parameter of the Lieb-Liniger model $\gamma = mg/\hbar^2\rho$, where g is the strength of the atom-atom interactions and $\rho = N/L$ is the linear density of the system. The excitations spectra are found by solving the appropriate integral equations (see [7] for details).

For higher temperatures, as it is expected, the model of noninteracting quasiparticles breaks and should be replaced by the one including interactions between quasiparticles. This is obviously not an easy task. However, a generalization of the Lieb-Liniger model was developed in the late 1960s which deals with the equilibrium thermodynamics of a one-dimensional system of bosons interacting via the delta function [4]. The following set of equations (units and symbols as in the original paper [4] are used throughout this paragraph),

$$\epsilon(k) = -\mu + k^2 - \frac{Tc}{\pi}\int_{-\infty}^{\infty}\frac{dq}{c^2+(k-q)^2} \times \ln\{1+\exp[\epsilon(q)/T]\}, \tag{2}$$

$$2\pi\rho(k)[1+\exp\epsilon(k)/T] = 1+2c\int_{-\infty}^{\infty}\frac{\rho(q)dq}{c^2+(k-q)^2}, \tag{3}$$

$$\frac{N}{L} = \int_{-\infty}^{\infty}\rho(k)dk, \tag{4}$$

allows us to find the excitation spectrum at any temperature. The integral equation (2) has to be solved for a given temperature T, the strength of the interaction c, and the chemical potential μ. The trial excitation spectrum $\epsilon(k)$ is used to get the density of excitations $\rho(k)$ from another integral equation (3). Finally, the condition (4) is checked. This is a self-consistent procedure resulting in the spectrum $\epsilon(k)$ which, as the authors of [4] suggest, might be regarded as the dispersion curve for a kind of effective, temperature-dependent, elementary excitations.

Below we follow this suggestion. In the limit of zero temperature Eqs. (2) and (3) reduce to the ones obtained earlier by Lieb [7]. The energy curve $\epsilon(k)$ is a monotonically increasing function crossing zero value at a point which is the maximal momentum for the Lieb excitations of type II. Part of excitation curve $\epsilon(k)$ for momenta smaller than this maximal value describes the energy of the second branch of excitations. The remaining part determines the dispersion relation of the modes of the first kind. After the splitting of the single Yang-Yang excitation spectrum into two excitation spectra by cutting at the maximal momentum $N\pi/L$, this momentum becomes zero momentum for each curve. Only then the zero-temperature limit can be recovered.

Now, at the thermal equilibrium in a given temperature T, the states of the gas of the quasiparticles are populated according to the probability distribution

$$P\left(\left\{n_{p_1}^{(1)},n_{p_2}^{(2)}\right\}\right) = \frac{1}{Z}e^{-\mathcal{E}\{n_{p_1}^{(1)},n_{p_2}^{(2)}\}/k_BT}, \tag{5}$$

where Z is the canonical partition function and the energy is obtained based on the Yang-Yang prescription. An efficient way to sample the phase space of the system of quasiparticles is to use a Monte Carlo method with the Metropolis algorithm [14]. Having a canonical ensemble of states one can find various statistical averages. In particular, in Fig. 1 we show an average number of the type II excitations for various temperatures for a system of $N = 1000$ atoms and $\gamma = 0.02$. We use $\hbar^2/mL^2k_B$ as the unit of temperature. As expected, the population of higher momentum, i.e., larger-energy quasiparticle states, gets larger at higher temperatures.

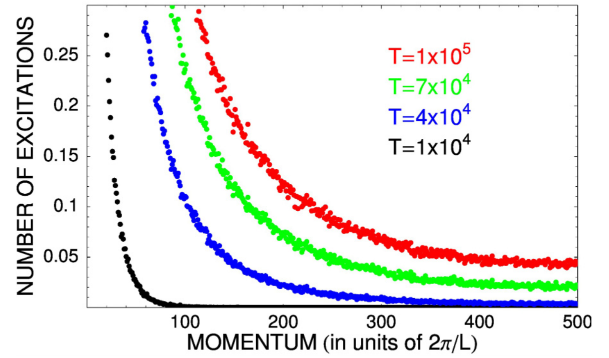

FIG. 1. (Color online) Average number of type II excitations as a function of quantized momentum at thermal equilibrium for the system of $N = 1000$ atoms and $\gamma = 0.02$ calculated within the Yang-Yang model. Dots correspond, from the top to bottom, to the temperatures 10^5, 7×10^4, 4×10^4, and 10^4 in units of $\hbar^2/mL^2k_B$. Maximal excitation momentum equals to $N\pi/L$. For the lowest considered temperature, $T = 10^4$, the Lieb-Liniger and Yang-Yang models give practically the same curves.

III. SOLITON-COUNTING PROCEDURE

To get the physical insight to the results presented in Fig. 1 we now turn to the approximate treatment of an interacting Bose gas, i.e., the classical field method [6]. Within this approach, which is an extension of the original Bogoliubov idea [15], the usual bosonic field operator $\hat{\Psi}(x)$ which annihilates an atom at point x is replaced by the complex wave function $\Psi(x)$. To study a thermal equilibrium state of a one-dimensional weakly interacting Bose gas we need an ensemble of classical fields $\Psi(x)$ corresponding to a given temperature. Such an ensemble can be obtained by using again the Monte Carlo algorithm [16].

In [17] we have already shown that dark solitons occur spontaneously in a one-dimensional interacting Bose gas at equilibrium. For further investigation we need a method allowing us to count dark solitons. Our reasoning is as follows. Let us start with a very simple approximation: we assume that the gas density is a sum of single regularized solitonic densities:

$$\rho(x,t) = \sum_j \rho_{\rm sol}(x - x_j - u_j t), \tag{6}$$

where x_j and u_j are the initial position and the velocity of the jth soliton, respectively. The density of a single dark soliton is given by [9,18]

$$\rho_{\rm sol}(x,u) = \rho_0[u^2 - 1 + (1-u^2)\tanh^2(x\sqrt{1-u^2})]. \tag{7}$$

Here and in the rest of this paragraph we use the healing length and the speed of sound as the units of length and velocity, respectively, and ρ_0 is the background density. In these units $|u| \leqslant 1$. The above approximation should be quite realistic when solitons are well separated, i.e., when the time they spend while colliding is a negligible fraction of the total evolution time. Now, the Fourier transform of the density Eq. (6), both in position and time variables, results in

$$\tilde{\rho}(k,\omega) = \sum_j e^{ikx_j} f(k,u_j)\,\delta(\omega + u_j k), \tag{8}$$

where the function $f(k,u) = \int e^{ikx}\rho_{\rm sol}(x)\,dx/2\pi$ is the Fourier transform of a single solitonic density which depends on the soliton velocity as a parameter. Since we rather use the numerical Fourier transform to calculate Eq. (8) one should keep in mind that, in fact, the Kronecker delta appears in Eq. (8) instead of the Dirac one. The square of the Fourier transform of the density Eq. (8) can be calculated as

$$|\tilde{\rho}(k,\omega)|^2 = \sum_j A_j |f(k,u_j)|^2 \, \delta(\omega + u_j k), \tag{9}$$

where the factor A_j equals

$$A_j = N_j + 2 \sum_{j_1 > j_2} \cos[k(x_{j_1} - x_{j_2})] \tag{10}$$

and N_j is the number of solitons having the velocity u_j (x_{j_n}'s are their initial positions). To get rid of the interference terms in Eq. (10) it is enough to average the formula (9) over many realizations since for thermal states the initial positions of solitons are random. Then we have

$$\langle |\tilde{\rho}(k,\omega)|^2 \rangle = \sum_j N_j \, |f(k,u_j)|^2 \, \delta(\omega + u_j k). \tag{11}$$

The prescription outlined above works as follows. First, for a given temperature we prepare an ensemble of classical fields. Next, each classical field is propagated according to the Gross-Pitaevskii equation of motion. For each realization, the square of the Fourier transform of the probability distribution in position and time is calculated. Finally, an average over realizations is performed. According to the formula (11) each piece of this data grouped around a particular value of ω carries information about the number of solitons having the velocity equal to $-\omega/k$. Therefore, dividing $\langle |\tilde{\rho}(k,\omega)|^2 \rangle$ by $|f(k,u)|^2$, where $u = -\omega/k$, gives us the number of solitons with the velocity u.

Before we start to analyze the thermal gas let us check how the procedure described above works in the case of the system consisting of a few solitons only. For that we propagate first the Gross-Pitaevskii equation of motion for two solitons having opposite velocities. The solitons move in free space with periodic boundary conditions. They must have opposite (but of the same magnitude) velocities to fulfill the periodic boundary conditions for the phase of the solitonic wave function. In Fig. 2

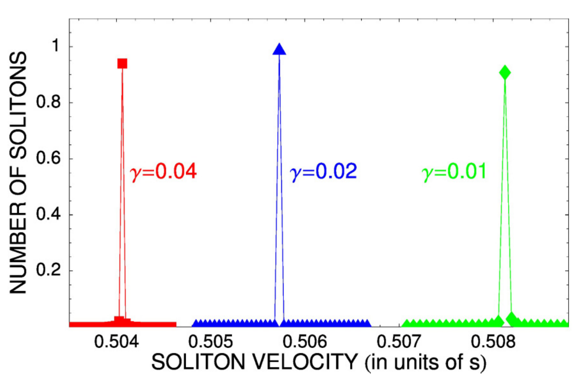

FIG. 2. (Color online) Number of solitons moving to the right as returned by the soliton-counting procedure for a system consisting of two solitons having velocities $u = \pm 0.5$ (in units of speed of sound, s). The strength of the nonlinear term in the nonlinear Schrödinger equation describing the evolution of solitons equals the strength of the term describing the interactions in the mean-field approximation for the system of $N = 1000$ atoms and $\gamma = 0.04$ (red squares), $\gamma = 0.02$ (blue triangles), and $\gamma = 0.01$ (green diamonds). The results were obtained based on the cut for $k = 10 \times 2\pi/L$.

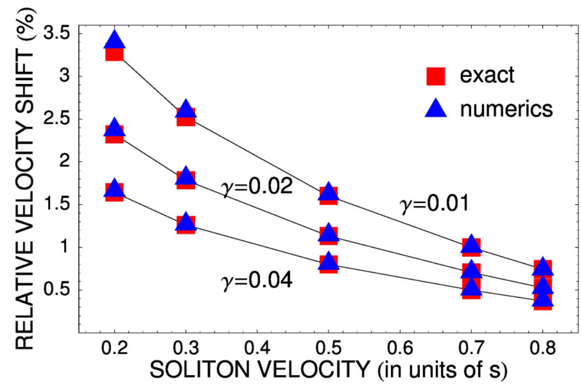

FIG. 3. (Color online) Relative velocity shift as returned by the soliton-counting procedure for a system consisting of two solitons as a function of soliton velocity. Different nonlinearity strengths $\gamma =$ 0.04, 0.02, and 0.01 are considered. Red squares (blue triangles) were obtained by analytical (numerical) calculations.

we plot the number of solitons as a function of velocity for various strengths of nonlinearity as a result of our procedure. The velocities of solitons equal $u = \pm 0.5$. However, as can be seen from Fig. 2, the procedure returns actually higher solitonic velocities and the shift of the velocity depends on the strength of the nonlinear term. This can be easily understood. The position of the soliton having velocity u after time t equals ut only when it moves freely. In the presence of another soliton with the velocity u', due to collisions, this position becomes

$$ut + \frac{t}{\frac{L}{|u-u'|}} \Delta x, \tag{12}$$

where Δx is the asymptotic phase shift for the binary solitonic collision [19] and the factor in front of it determines the number of such collisions. Therefore, the real position of the peak corresponding to the soliton of the velocity u should be at

$$u\left(1 + \frac{|u-u'|}{u}\frac{\Delta x}{L}\right). \tag{13}$$

It is indeed, as shown in Fig. 3 for $\gamma = 0.02$ as well as for twice smaller and twice larger nonlinearity strength. Relative velocity shifts calculated based on the analytical formula for the phase shift [19] are marked by red squares. Excellent agreement with numerical data (blue triangles) is clear.

By adding the numbers within single peaks in Fig. 2 one gets values very close to 1. This means that for the system including two solitons the soliton-counting procedure works well, giving the expected result—two solitons are present, one moving to the right and the other moving to the left. However, the picture becomes more complex when the number of solitons increases. Then we have more collisions between solitons. This results in larger velocity shifts. But also the simple approximation made by us as regards the total density Eq. (6) is no longer valid—collisions are frequent and solitons overlap each other a significant fraction of the evolution time. Figure 4 compares the number of solitons calculated via the soliton-counting procedure for the systems consisting of up to 14 solitons. It is clear from Fig. 4 that the number of solitons returned by the soliton-counting procedure is lowered when collisions between solitons become important

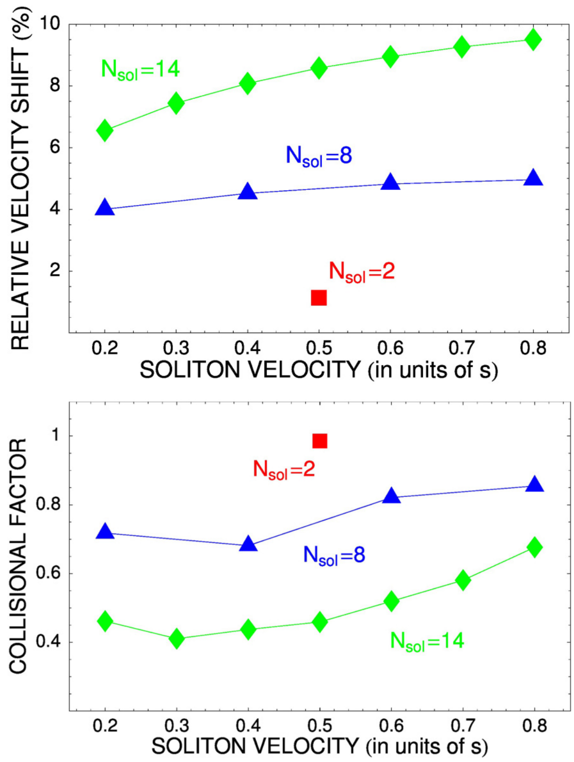

FIG. 4. (Color online) Relative velocity shift (upper frame) and the number of solitons moving to the right (lower frame) as returned by the soliton-counting procedure for the system consisting of 2 (red square), 8 (blue triangles), and 14 (green diamonds) solitons. Here, $\gamma = 0.02$.

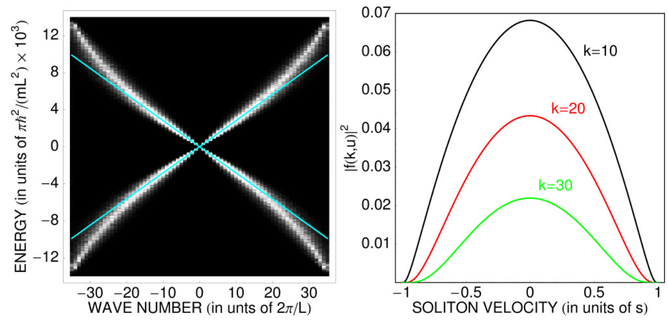

FIG. 5. (Color online) Fourier transform, in the position and time, of the density of the classical field, first squared and next averaged over 12 realizations (left panel). The number of atoms $N = 1000$ and the temperature is 10^4. Blue solid lines are drawn according to $\omega = \pm sk$, where s is the speed of sound. The right panel shows $|f(k,u)|^2$ for $k = 10 \times 2\pi/L$, $20 \times 2\pi/L$, and $30 \times 2\pi/L$.

or dominant. Since the number of solitons in a thermal state increases with temperature one should expect that the counting procedure in the simple version discussed above will start to return numbers lower than exact ones at some temperatures.

IV. SOLITONS IN THE THERMAL EQUILIBRIUM STATE

Having in mind all the limitations of the soliton-counting procedure related to the density ansatz Eq. (6), we now go back to the case of a gas in thermal equilibrium. In Fig. 5 we show an average over 12 realizations, the left-hand side of Eq. (11), for the system of $N = 1000$ rubidium atoms at the temperature 10^4. The superimposed blue lines are drawn as $\omega = \pm sk$, where s is the speed of sound. To get an average number of dark solitons excited in a gas we need to investigate an area within the sonic cone. We consider the cut of $\langle|\tilde{\rho}(k,\omega)|^2\rangle$ along a constant value of k, in our case $k = 10 \times 2\pi/L$. Next, we divide $\langle|\tilde{\rho}(k,\omega)|^2\rangle$ by $|f(k,u)|^2$, where $u = -\omega/k$, getting the number of solitons with the velocity u. Now, introducing a momentum of a soliton (defined as the mean value of the momentum operator calculated in the soliton state) one can relate the soliton momentum, $p(u)$, with its velocity as $p(u) = 2\,(\arccos u - u\sqrt{1-u^2})$ [9,12]. In this way we obtain the information about the spectrum of momenta of solitons which are present in a gas at thermal equilibrium. Finally, remembering that according to the Lieb model [7] available momenta are quantized, we count solitons by grouping their momenta around these equally spaced quantized values.

We show the average number of dark solitons as a function of momentum in Fig. 6 for the system consisting of $N = 1000$ weakly interacting ($\gamma = 0.02$) atoms at the temperatures $T = 10^4$ (gray line) and $T = 4 \times 10^4$ (light red line). In the same figure we compare our results for dark solitons with those for the Yang-Yang model type II excitations (black line for $T = 10^4$ and dark red line for $T = 4 \times 10^4$). The agreement is good for lower temperature, which is a strong indication that the Yang-Yang (as well as the Lieb-Liniger) type II modes might be the quantum solitons. However, for higher temperatures both approaches start to differ. The average number of dark solitons gets lower than the number of type II excitations. This is not surprising since for higher temperatures collisions between solitons become important and the soliton-counting procedure underestimates the number of solitons (see Fig. 4, lower frame). This can be easily corrected

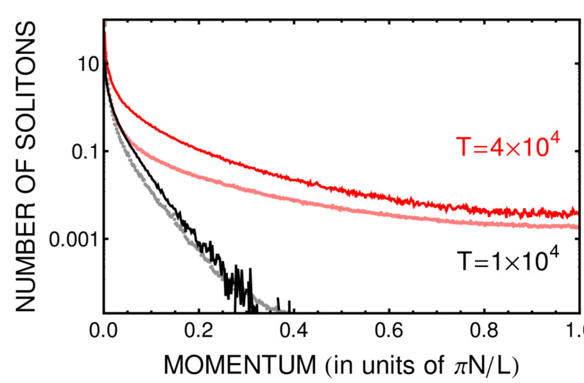

FIG. 6. (Color online) Comparison between the Yang-Yang model and the classical field approximation for the temperatures $T = 10^4$ and 4×10^4. Frame shows the average number of type II excitations (from the Yang-Yang model) and of dark solitons (from the classical field approximation) vs the momentum. The data are additionally averaged over the direction of the momentum. For the lowest considered temperature, $T = 10^4$, the agreement between numerical (dark line) and analytical (black line) results is good. For higher temperature, $T = 4 \times 10^4$, a disagreement appears.

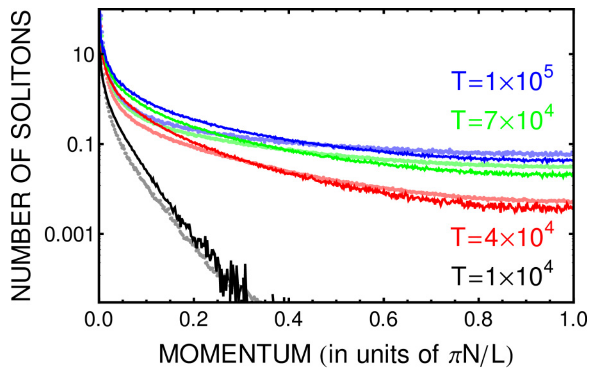

FIG. 7. (Color online) Comparison between the Yang-Yang model and the classical field approximation for the temperatures (from the top to bottom) 10^5, 7×10^4, 4×10^4, and 10^4. Frame shows the average number of type II excitations (from the Yang-Yang model: dark lines) and of dark solitons (from the classical field approximation and soliton-counting procedure corrected for higher temperatures due to the presence of solitonic collisions: light lines) vs the momentum. The data are additionally averaged over the direction of the momentum. Now, the agreement between numerical and analytical results is partially restored for all temperatures. The quantitative disagreement for higher temperatures is still present because, in fact, both the relative velocity shift and the collisional factor slightly depend on the solitonic velocity (see Fig. 4) and two-parameter correction is insufficient.

assuming that both the relative velocity shift and the collisional defect do not depend on the soliton velocity (see Fig. 4). Then by choosing phenomenologically two parameters one can first move soliton velocities toward lower values and second increase the number of solitons for a given velocity. The results of such a phenomenological treatment are shown in Fig. 7.

V. SUMMARY

In summary, we have studied a uniform one-dimensional weakly interacting Bose gas at a thermal equilibrium. Properties of such a system can be investigated in terms of a gas of two families of quasiparticles introduced originally by Lieb [7]. For low temperatures the ideal gas approximation works well. For higher temperatures, however, quasiparticles become strongly interacting and new noninteracting effective excitations should be used according to the Yang-Yang prescription [4]. In this way we found statistical distributions of excitations of the second type for nonzero temperatures. On the other hand, employing the classical field approximation to a uniform interacting Bose gas at thermal equilibrium we determined an average number of dark solitons for a given momentum. Both distributions match, which proves that type II excitations are indeed quantum solitons.

ACKNOWLEDGMENTS

This work was supported by the National Science Center (Poland) Grants No. DEC-2011/01/B/ST2/05125 (T.K. and M.G.) and No. DEC-2012/04/A/ST2/00090 (M.B., T.S., and K.R.). The Centre for Quantum Technologies is a Research Centre of Excellence funded by the Ministry of Education and the National Research Foundation of Singapore.

[1] E. H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).
[2] J. Sato, R. Kanamoto, E. Kaminishi, and T. Deguchi, Phys. Rev. Lett. **108**, 110401 (2012).
[3] R. Kanamoto, L. D. Carr, and M. Ueda, Phys. Rev. A **81**, 023625 (2010).
[4] C. N. Yang and C. P. Yang, J. Math. Phys. **10**, 1115 (1969).
[5] K. Góral, M. Gajda, and K. Rzążewski, Opt. Express **8**, 92 (2001).
[6] M. Brewczyk, M. Gajda, and K. Rzążewski, J. Phys. B **40**, R1 (2007).
[7] E. H. Lieb, Phys. Rev. **130**, 1616 (1963).
[8] M. Brewczyk, P. Borowski, M. Gajda, and K. Rzążewski, J. Phys. B **37**, 2725 (2004).
[9] P. P. Kulish, S. V. Manakov, and L. D. Faddeev, Theor. Math. Phys. **28**, 615 (1976).
[10] M. Ishikawa and H. Takayama, J. Phys. Soc. Jpn. **49**, 1242 (1980).
[11] E. B. Kolomeisky, T. J. Newman, J. P. Straley, and X. Qi, Phys. Rev. Lett. **85**, 1146 (2000).
[12] A. D. Jackson and G. M. Kavoulakis, Phys. Rev. Lett. **89**, 070403 (2002).
[13] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press, Oxford, 1969).
[14] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, J. Chem. Phys. **21**, 1087 (1953).
[15] N. N. Bogoliubov, J. Phys. (Moscow) **11**, 23 (1947).
[16] E. Witkowska, M. Gajda, and K. Rzążewski, Opt. Commun. **283**, 671 (2010).
[17] T. Karpiuk, P. Deuar, P. Bienias, E. Witkowska, K. Pawłowski, M. Gajda, K. Rzążewski, and M. Brewczyk, Phys. Rev. Lett. **109**, 205302 (2012).
[18] V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **64**, 1627 (1973) [Sov. Phys. JETP **37**, 823 (1973)].
[19] H. Cai, F.-M. Liu, and N.-N. Huang, Int. J. Theor. Phys. **44**, 255 (2005).