

Ground-state entanglement of spin-1 bosons undergoing superexchange interactions in optical superlattices

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Received April 1, 2014; revised May 29, 2014; accepted June 8, 2014;
posted June 9, 2014 (Doc. ID 209378); published July 15, 2014

We will discuss a model with ultracold atoms confined in optical superlattices. In particular, we will study the ground-state properties of two spin-1 bosons trapped in a double-well potential. Depending on the external magnetic field and biquadratic interactions, different phases of magnetic order are realized. Applying von Neumann entropy and the number of relevant orbitals, we will quantify the bipartite entanglement between particles. Changing the values of the parameters determining the superlattices, we can switch the system between differently entangled states. © 2014 Optical Society of America

OCIS codes: (020.7010) Laser trapping; (270.5585) Quantum information and processing.
<http://dx.doi.org/10.1364/JOSAB.31.001845>

1. INTRODUCTION

Quantum entanglement is one of the fundamental concepts of physics, and it plays a crucial role in quantum information theory [1]. Without the idea of entanglement, it is not possible to understand the many spectacular properties of quantum systems or phenomena related to them. At this point, one can mention teleportation [2–6], quantum cryptography [7,8], topological order [9,10], quantumness of physical systems [11], etc. In the context of strongly correlated many-body systems, together with quantum statistics, the entanglement between indistinguishable particles is responsible for the non-intuitive properties of macroscopic systems [12]. The relation between ground-state entanglement and quantum phase transitions provides a bridge between quantum information theory and condensed-matter physics [13–15]. For instance, transitions from product states to strongly entangled ones in a two-qubit Heisenberg nuclear spin chain have been observed experimentally in systems affected by varying the external magnetic field [16,17].

In recent years, a wide perspective for experimental realization of spin chain systems has been found in models involving ultracold atoms trapped in optical lattices [18,19], ultracold quantum gates [20], coupled quantum dots [21], etc. Contrary to solid-state physics, ultracold atomic systems can be quite well tuned to situations in which their properties are well described by simplified models of condensed-matter theory. Moreover, they permit engineering and manipulating quantum states in single lattice sites as well as collectively in the whole lattice. It is possible to tune all relevant parameters almost adiabatically, even in the vicinity of the phase transitions [22,23], and, as a consequence, to test scaling invariance and universality [24]. Such systems have also opened completely new possibilities for answering fundamental questions of quantum information theory related to transitions from

product to entangled states, quantum state transfers, quantum magnetism, spin dynamics, etc. [25–29]. In this way, ultracold atomic systems have become dedicated quantum simulators [30,31] for condensed-matter problems, where the fundamental parameters of theoretical models can be controlled experimentally.

The properties of interacting ultracold atoms confined in optical lattices are typically described in the language of Hubbard-like models. In the simplest case of spinless bosons interacting repulsively via short-range delta-like interactions, the Bose–Hubbard model has only two competing terms related to single-particle tunneling to the neighboring site and the additional energy cost when two bosons meet in a given site [32]. Extended Hubbard models (for fermions and bosons) originate in taking into account higher bands of optical lattices, long-range interactions, and the internal structure of interacting particles [33–38]. In particular, in scenarios when tunneling is dominated by interactions, all Hubbard models can be simplified perturbatively and can be rewritten to appropriate lattice-spin models. For example, when the standard Bose–Hubbard model is considered and the system remains in the Mott-insulator phase, one can find the most relevant corrections by treating the tunneling as a virtual process in the second order of perturbation theory [39]. With this observation, Simon *et al.* experimentally mimicked the one-dimensional Ising model [40]. In a more general case, when the internal structure of ultracold atoms cannot be neglected, the low-energy effective Hamiltonian requires superexchange interactions [41]. Then, the corresponding spin model also includes higher powers of the scalar products of spin operators at neighboring sites, $H = \sum_k J_k (\hat{S}_i \cdot \hat{S}_j)^k$ [42–46]. For example, for the case of spin-1 bosons, the system is described by the Heisenberg model with an additional biquadratic term, and is called the quadratic–biquadratic Heisenberg (QBH) model

[42,43,47]. So far, mostly linear Heisenberg spin models have been studied intensively over a wide range of parameters [48–53]. The QBH model has attracted great interest mainly due to a wealth of possible quantum phases, predicted in its antiferromagnetic case, and in, for instance, Haldane, nematic, dimerized, and trimerized phases [54–56], as well. The QBH model was also applied successfully in studies concerning magnetic properties and energy level distribution in several real materials [57–61], where the validity of the biquadratic interaction has been emphasized.

Motivated by all these observations, we shall discuss here the properties of spin-1 bosons (for example, the alkali atoms of 23-sodium or 87-rubidium) confined in an optical superlattice, i.e., a lattice created by the interference of two independent laser standing waves with a commensurate frequency ratio equal to 2. Such a lattice is characterized by a structure involving weakly coupled double-well potentials and, therefore, it could be quite a good arena for studying two-site Hubbard models. Moreover, in the limit of strong repulsions, this model can be simplified to the two-site QBH model [42,43,47]. We should note that in [62] the authors have concentrated on the effect of the asymmetry of the double-well potential in the realization of various types of magnetic order. They have also discussed some aspects related to the bipartite entanglement between sites in a quite limited regime of parameters. Here, we complement and extend these studies by assuming that a full range of parameters of the resulting Hamiltonian can be reached experimentally. It can be done directly (for instance, by using the optical Feshbach resonances [54,63]) or by performing some kind of “quantum simulation” where the system is prepared in its excited state (as has been proposed [47,55]). In particular, our main goal is to analyze the influence of the biquadratic interaction on the ground-state entanglement.

The paper is organized as follows: in Section 2 we introduce the model studied here and present its theoretical background. In Section 3 we give a full analysis of the spectrum of the two-site Hamiltonian and in Section 4 we present the phase diagrams of the ground state in a whole accessible range of parameters. Section 5 is devoted to the presentation of the main results related to the entanglement generation in the ground state. Finally, we present our conclusions in Section 6.

2. TWO-SITE SPIN-MODEL HAMILTONIAN

In this paper, we consider spin-1 ultracold bosons confined in a one-dimensional superlattice potential. We assume that the dynamics is frozen in two perpendicular directions and the particles remain in the ground state of the confining potential. Moreover, laser beams are configured in such a way that one can treat the system as independent double-well potentials. In other words, we assume that the tunneling amplitudes between potential dimers are much smaller than the tunnelings inside the dimers. Such a situation is well described with the two-site Bose–Hubbard model of the form

$$H = -t(\hat{a}_{L\sigma}^\dagger \hat{a}_{R\sigma} + \hat{a}_{R\sigma}^\dagger \hat{a}_{L\sigma}) + \frac{U_0}{2} \sum_{i=L,R} \hat{n}_i(\hat{n}_i - 1) + \frac{U_2}{2} \sum_{i=L,R} (\hat{S}_i^2 - 2\hat{n}_i) - \gamma \mathbf{B} \cdot (\hat{\mathbf{S}}_L + \hat{\mathbf{S}}_R), \quad (1)$$

where $\hat{a}_{i\sigma}$ annihilates a boson in spin state $\sigma \in \{-1, 0, 1\}$ at the left (L) or right (R) well; $\hat{n}_i = \sum_{\sigma} \hat{a}_{i\sigma}^\dagger \hat{a}_{i\sigma}$ is the total number of particles at a given site; and \hat{S}_L and \hat{S}_R are total spin operators. The parameters U_0 and U_2 describe the repulsive interaction between two particles confined in one lattice site. They are proportional to the s-wave scattering lengths a_0 and a_2 when the total spins of colliding particles are $S = 0$ and $S = 2$, respectively (see, for example, [43]). The last term in the Hamiltonian (1) describes the linear Zeeman effect in the external magnetic field \mathbf{B} . The parameter $\gamma = g\mu_B$, where μ_B is the Bohr magneton and g is the Landé factor. In further analysis, we will assume that the external magnetic field is oriented along z axis. It is worth noticing that the inversion of the spin-axis quantization is equivalent to the inversion of the magnetic field. Therefore, without losing generality, one can assume that magnetic field B^z is not negative. All predictions for $B^z < 0$ have their counterparts with the opposite sign of the spins in $B^z > 0$. In fact, the Hamiltonian (1) is an effective Hamiltonian, general enough to be applied in the description of a large family of physical systems. For instance, it was used in the discussion of various interesting phenomena appearing in Kerr-like quantum-optical or nanosystem models, such as photon (phonon) blockade [64–66] (sometimes called nonlinear quantum scissors [67–70]) or quantum-chaotic behavior [71–75].

Our aim is to study the system with unit filling in the strongly repulsive regime, i.e., when the nonlocal tunneling term in the Hamiltonian (1) can be treated as a small perturbation (when compared with the sum of local interaction terms). Therefore, we rewrite the Hamiltonian as the effective one in the second order of perturbation in t [42,43]:

$$H = J_0 + J_1 \hat{S}_L \cdot \hat{S}_R + J_2 (\hat{S}_L \cdot \hat{S}_R)^2 - \gamma B^z (\hat{S}_L^z + \hat{S}_R^z), \quad (2)$$

where $J_1 = -(2t^2/U_2)$, $J_2 = -(2t^2/3U_2) - (4t^2/3U_0)$, and $J_0 = J_1 - J_2$.

Since we shall study the effects of the biquadratic term in the Hamiltonian (2), it is convenient to parameterize the Hamiltonian (2) with three dimensionless parameters:

$$H = \lambda \hat{S}_L \cdot \hat{S}_R + \tan \theta (\hat{S}_L \cdot \hat{S}_R)^2 - h (\hat{S}_L^z + \hat{S}_R^z), \quad (3)$$

where $\lambda = J_1/|J_1| = \pm 1$ is a sign of a linear term, $\tan \theta = J_2/|J_1|$, and $h = \gamma B^z/|J_1|$. For convenience, we also set the energy scale in such a way that $J_0 = 0$. A positive (negative) λ favors antiferromagnetic (ferromagnetic) orientation of spins. It should be emphasized that although, from the model point of view, the angle θ lies within the interval $(-\pi/2, \pi/2)$ for the system of ultracold atoms when the ratio is typically $U_2/U_0 > 0$, the angle θ cannot cover this range completely [54]. Nevertheless, an experimental scheme where a whole range of θ can be obtained was also proposed [47,55]. Moreover, in contrast to usual condensed-matter systems (see, for example, [41]), ultracold atoms make it possible to engineer experimental setups where biquadratic coupling is larger than the linear one.

It should also be underlined that the effective Hamiltonian (3) can be derived from the more general Hubbard-like Hamiltonian (1) only for repulsive interactions. Only for this case, the Mott-insulator phase with one particle in each lattice site is a true ground state of the system, in the limit of

vanishing tunneling. This means that only the negative sign of λ can be obtained in this framework. However, it was shown recently that a positive λ can be engineered effectively by preparing the system in its excited state [47,55]. As a consequence, although such a system is not in the true ground state, it can be quite well described with the use of the effective Hamiltonian (3) with a positive λ and for long time scales.

3. SPECTRUM OF THE EFFECTIVE HAMILTONIAN

The Hamiltonian (3) commutes with the total spin operator $\hat{S}^z = \hat{S}_L^z + \hat{S}_R^z$. Therefore, it can be diagonalized in the subspaces of a given spin. All these subspaces are spanned together by nine natural eigenvectors of total spin $|\sigma, \sigma'\rangle = \hat{a}_{L\sigma}^\dagger \hat{a}_{R\sigma'}^\dagger |\text{vac}\rangle$. Performing exact diagonalization of the Hamiltonian, one can find a local ground state for each subspace, corresponding to a given spin. As a consequence, the true ground state of the system is the one with the lowest energy. It is quite obvious that the states $|1, 1\rangle$ and $|-1, -1\rangle$ with the largest absolute total spin ($S^z = \pm 2$) are the eigenstates of the Hamiltonian (3) for any set of dimensionless parameters. It comes directly from the fact that they are only states with the largest (smallest) total spin. Their eigenenergies are equal to $E_{\pm 2} = \lambda + \tan \theta \mp 2h$. For these states, we observe no entanglement between two sites.

Subspaces with total spin $S^z = \pm 1$ are spanned by two natural eigenvectors $|1, 0\rangle$ and $|0, 1\rangle$ for $S^z = 1$ and $|-1, 0\rangle$ and $|0, -1\rangle$ for $S^z = -1$. Due to the presence of additional Z_2 symmetry of the Hamiltonian (3), namely, the mirror left–right symmetry, the eigenstates of the Hamiltonian in these subspaces are also easy to find and they are represented by two maximally entangled states (MESs)—Bell states. For $S^z = 1$, these states are

$$||1; \pm\rangle\rangle = \frac{|0, 1\rangle \pm |1, 0\rangle}{\sqrt{2}} \quad (4a)$$

with corresponding eigenenergies $E_{1;\pm} = \pm(\lambda + \tan \theta) - h$. Similarly, for $S^z = -1$ these states are

$$||-1; \pm\rangle\rangle = \frac{|0, -1\rangle \pm |-1, 0\rangle}{\sqrt{2}} \quad (4b)$$

with corresponding eigenenergies $E_{-1;\pm} = \pm(\lambda + \tan \theta) + h$. The most interesting situation is in the remaining subspace of total spin $S^z = 0$. This subspace is spanned by three natural eigenvectors $|-1, 1\rangle$, $|0, 0\rangle$, and $|1, -1\rangle$. Due to the above-mentioned mirror symmetry of the Hamiltonian, three eigenstates of the Hamiltonian in this subspace have the following forms:

$$||0; 0\rangle\rangle = \frac{|-1, 1\rangle - |1, -1\rangle}{\sqrt{2}}, \quad (5a)$$

$$||0; \pm\rangle\rangle = \cos \alpha_{\pm} \frac{|-1, 1\rangle + |1, -1\rangle}{\sqrt{2}} \pm \sin \alpha_{\pm} |0, 0\rangle, \quad (5b)$$

with corresponding eigenenergies

$$E_{0,0} = -\lambda + \tan \theta, \quad (6a)$$

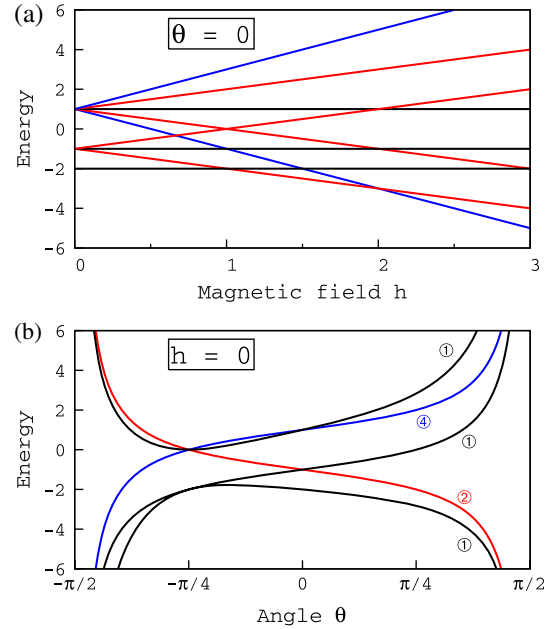


Fig. 1. Spectrum of the Hamiltonian (3) for $\lambda = 1$ in two limiting situations: (a) vanishing biquadratic interaction $\theta = 0$ and (b) vanishing external magnetic field $h = 0$. In (a), different colors correspond to states with different total spin S^z ($S^z = 0$, black line; $S^z = \pm 1$, red line; $S^z = \pm 2$, blue line). In (b), the circled numbers denote the degree of degeneracy of the states.

$$E_{0;\pm} = (-\lambda + \tan \theta \pm \delta)/2, \quad (6b)$$

where $\delta = [9(\lambda + \tan \theta)^2 - 4\lambda \tan \theta]^{1/2}$. The mixing angles α_{\pm} are related to the parameters of the Hamiltonian by the condition $\cos \alpha_{\pm} = \sqrt{|E_{0;\pm}|/\delta}$. It can be checked easily that the state $||0; -\rangle\rangle$ is a ground state in the considered subspace for any set of parameters of the Hamiltonian. In addition, for the particular set of parameters $\lambda = 1$ and $\theta_0 = -\pi/4$, it is degenerated with the state $||0; 0\rangle\rangle$. As will be explained in Section 5, at this point the state $||0; -\rangle\rangle$ changes its character from a three-dimensional triplet (for $\theta > \theta_0$) to a three-dimensional singlet (for $\theta < \theta_0$) Bell state.

The whole spectrum of the Hamiltonian for two cases, vanishing biquadratic interaction ($\theta = 0$) and vanishing external magnetic field ($h = 0$), when a positive linear interaction is assumed ($\lambda = 1$), is plotted in Fig. 1. As we can see, the presence of the biquadratic term does not lift the degeneracy completely (the remaining multiplicity of energy levels is marked by numbers in circles).

4. GROUND-STATE PHASE DIAGRAM

To get a clearer picture of possible scenarios, we present in Fig. 2 the ground-state phase diagram of the system. This diagram is obtained with the application of exact diagonalization of the Hamiltonian (3). We see that for the given values of the parameters describing the Hamiltonian and magnetic field $h > 0$, the true ground state of the system belongs to the one of three subspaces of total spin $S^z = 0, 1, 2$. Obviously, for a negative magnetic field $h < 0$, the phase diagram is exactly the same, but quantum phases correspond to the opposite sign of the total spin.

As can be seen in Fig. 2, the phase diagram crucially depends on the sign of the linear coupling λ . It is worth noticing

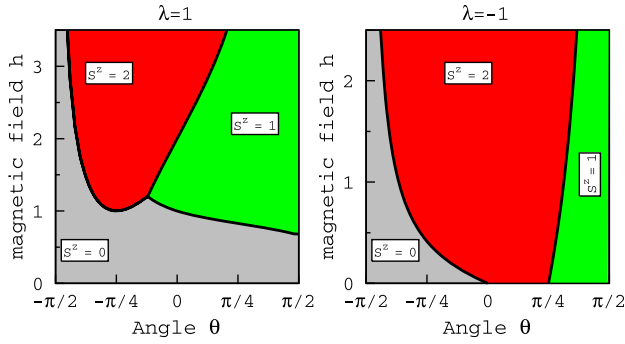


Fig. 2. Ground-state phase diagram of the system. For a given angle θ and a magnetic field h , the ground state of the system belongs to the one of three subspaces characterized by total spin S^z . Note that the properties of the ground state depend highly on the sign of the linear coupling term λ . The phase diagram for negative magnetic field h has identical phases, but with opposite total spin.

that for both the cases $\lambda = \pm 1$, the ferrimagnetic-like quantum phase with total spin $S^z = 1$ is the only phase that is confined by some minimal values of the biquadratic interaction. However, one should keep in mind that the ground state is always in the ferromagnetic phase (with the largest total spin $S^z = 2$) for large enough values of magnetic field, whereas the most interesting antiferromagnetic-like phase with total spin $S^z = 0$ can be reached only for sufficiently small values of magnetic field. When, additionally, λ is positive, this phase ($S^z = 0$) can be reached for any value of biquadratic interactions. However, when λ is negative, a negative biquadratic interaction is also needed to reach the antiferromagnetic-like phase.

Due to the fact that in the Hamiltonian (3) there is no term corresponding to coupling states with different S^z , the phases are separated by sharp borders. All quantum phase transitions appearing in the system are of first order—they are characterized by discontinuities in the first derivative of the ground-state energy. In other words, at the transition points the ground state of the system “jumps” from one subspace of total spin to the other one. For example, when $\lambda = 1$ and $\theta = 0$, we see that the system undergoes two transitions for $h = 1$ and $h = 2$. These transitions are directly related to the crossings of the energy levels shown in Fig. 1(b), and at those points, the magnetization of the ground state changes its value suddenly.

5. QUANTUM CORRELATIONS IN THE GROUND STATE

In this section, we will discuss the entanglement properties of the ground state more deeply. To study them we shall use two quantities to determine the entanglement present in the system. One of them is the following correlation parameter, introduced in [76]:

$$\mathcal{K}(\hat{\rho}_L) = \left(\sum_i \eta_i^2 \right)^{-1}, \quad (7)$$

where η_i are the eigenvalues of the left-site reduced density matrix $\hat{\rho}_L = \text{Tr}_R |\mathcal{G}\rangle\langle\mathcal{G}|$ obtained from the full, two-site ground-state density matrix $|\mathcal{G}\rangle\langle\mathcal{G}|$ by tracing out the degrees of freedom of the right well (obviously this parameter can be derived for the right site without changing the results). The parameter \mathcal{K} gives an effective number of single-particle orbitals occupied in the given many-body state. In particular,

when the one-site density matrix has n equal eigenvalues, then $\mathcal{K} = n$. Additionally, when we are dealing with product states, $\mathcal{K} = 1$. This parameter (\mathcal{K}) can be very useful in studies of bipartite systems [77].

Another measure of the system entanglement that we shall apply in this paper is von Neumann entropy. This measure is commonly used in numerous papers and can be defined as

$$S(\hat{\rho}_L) = \text{Tr}(\hat{\rho}_L \log_2 \hat{\rho}_L) = \sum_i \eta_i \log_2 \eta_i. \quad (8)$$

This entropy is even more interesting than the number of relevant orbitals \mathcal{K} since it is directly related to the properties of the system in the thermodynamic limit, as well as in the quantum information context. The entropy defined in Eq. (8) for a bipartite system ranges from 0 for completely disentangled (product) states to $\log_2 D$ for MESs defined in D -dimensional Hilbert space. On the other hand, we can discuss our system in terms of quantum information theory. For such a case, we treat the system as that of two D -dimensional qudits [78]. Therefore, besides the total spin used previously, the phases can be distinguished explicitly with respect to the quantum state of the system and consequently, the dimension of the Hilbert subspaces in which these states are defined. Therefore, the system found in a given phase can be represented by the quantum information theory objects corresponding to it. For instance, the bipartite system in the antiferromagnetic-like phase is described by the state defined in three-dimensional Hilbert subspace and hence, can be treated as a qutrit–qutrit system. Similarly, the system in the ferrimagnetic-like phase can be considered as a qubit–qubit one.

The quantities \mathcal{K} and \mathcal{S} characterize the global properties of the shared entanglement and they can be insensitive to the internal structure of the state. For example, there are various MESs that are characterized by the same \mathcal{K} and \mathcal{S} . Since we are interested in two particular generalized Bell states [79],

$$|3, \mathcal{B}_S\rangle = \frac{1}{\sqrt{3}}(|1, -1\rangle + |-1, 1\rangle - |0, 0\rangle), \quad (9a)$$

$$|3, \mathcal{B}_T\rangle = \frac{1}{\sqrt{3}}(|1, -1\rangle + |-1, 1\rangle + |0, 0\rangle), \quad (9b)$$

to overcome this problem, we shall measure the relative distance between the ground state of the system $|\mathcal{G}\rangle$ and these Bell states. Such relative distances are represented by the fidelities $\mathcal{F}_S = |\langle 3, \mathcal{B}_S | \mathcal{G} \rangle|^2$ and $\mathcal{F}_T = |\langle 3, \mathcal{B}_T | \mathcal{G} \rangle|^2$, respectively.

Note that, by definition, these states are not orthogonal. As a consequence, even if the fidelity calculated with respect to one of them is equal to 1, the remaining one is not zero [see Fig. 3(c)]. Thus, only simultaneous inspection of all three parameters (\mathcal{S} , \mathcal{K} , and $\mathcal{F}_{T,S}$) can give an insight into the physical situation discussed.

A. Positive Linear Coupling ($\lambda = +1$)

We shall discuss various cases that are interesting from the point of view of the entanglement generation processes. One of the possible situations (the most trivial one) is that of total spin $S^z = \pm 2$. Simply, being in these phases the system does not exhibit any nonlocal correlations. For this case,

\mathcal{K} and S are equal to 1 and 0, respectively, and, therefore, the ground state is a product state.

Whenever the ground state of the system is in the ferrimagnetic-like phase (total spin $S^z = \pm 1$), the system remains in the singlet Bell state for a qubit–qubit system, $|1, -\rangle$, with $\mathcal{K} = 2$ and entropy $S = 1$. It is worth mentioning that in this phase the state $|1, -\rangle$ is the exact ground state for any values of the parameters appearing in the Hamiltonian. Therefore, the entanglement of the ground state in this phase is robust against any changes of these parameters.

The most interesting situation is in the antiferromagnetic-like phase with $S^z = 0$. For this case, the ground state of the system, and, as a consequence, the degree of the system entanglement, depends crucially on the mixing angle θ . In Fig. 3(a), entropy S and the number of relevant orbitals \mathcal{K} for this phase are plotted. They can be calculated analytically directly from Eq. (5b):

$$S = \frac{E_{0,-}}{\delta} \log_2 \left(\frac{|E_{0,-}|}{2\delta} \right) + \frac{2(\lambda + \tan \theta)^2}{\delta E_{0,-}} \log_2 \left(\frac{2(\lambda + \tan \theta)^2}{\delta |E_{0,-}|} \right), \quad (10a)$$

$$\mathcal{K} = \frac{2\delta^2 E_{0,-}^2}{8(\lambda + \tan \theta)^4 + E_{0,-}^4}. \quad (10b)$$

Both the quantities vary with the biquadratic interaction strength, and hence, different qutrit–qutrit MESs can be generated [Fig. 3(a)]. In Figs. 3(b) and 3(c), other quantities that allow characterizing fully the properties of the ground state $|G\rangle$ are shown. In particular, in Fig. 3(b) we have plotted the probabilities of finding the ground state in two-site product states, i.e., $p_{\pm} = |\langle \mp 1, \pm 1 | G \rangle|^2$ (blue dashed line) and $p_0 = |\langle 0, 0 | G \rangle|^2$ (red solid line). Finally, in Fig. 3(c) we present the fidelities \mathcal{F}_S and \mathcal{F}_T .

As shown in Fig. 3(a), looking at the strong repulsion biquadratic interaction regime ($\theta \rightarrow -\pi/2$), entropy $S = \log_2 3$, whereas the number of relevant orbitals $\mathcal{K} = 3$. For such a case, the ground state of the system is the maximally entangled triplet state:

$$|G\rangle \xrightarrow{\theta \rightarrow -\pi/2} |3, B_T\rangle. \quad (11)$$

When the biquadratic interaction grows, both S and \mathcal{K} start decreasing rapidly. Thus, for $\theta_0 = -\pi/4$, entropy $S = \log_2 2$ and $\mathcal{K} = 2$, which is related to vanishing of the probability p_0 . As a consequence, at this point the ground state, initially specified in $3 \otimes 3$ -dimensional Hilbert space, reduces to the state defined in $2 \otimes 2$ subspace. This state can be written as

$$|2, B_T\rangle = \frac{1}{\sqrt{2}}(|1, -1\rangle + |-1, 1\rangle). \quad (12)$$

In fact, it is the Bell MES defined in $2 \otimes 2$ Hilbert space, and here we are dealing with a qubit–qubit system. Moreover, one should remember that for $\theta = \theta_0$ the ground state $|2, B_T\rangle$ is degenerated with the other Bell state $|0; 0\rangle$.

When $\theta > \theta_0$, the probability amplitude $\langle 0, 0 | G \rangle$ changes its sign and, as a consequence, the ground state changes its nature, switching from a triplet- to a singlet-like state (they are not perfect triplet and singlet states, as some amount of

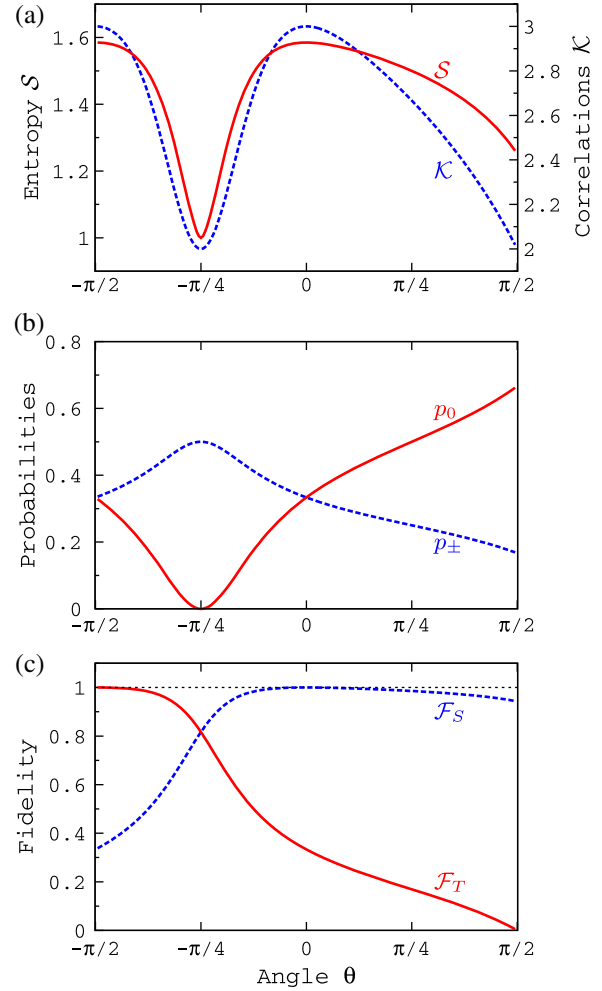


Fig. 3. Ground-state properties for positive linear coupling λ in the phase with total spin $S^z = 0$ as a function of mixing angle θ . (a) Number of relevant orbitals \mathcal{K} (blue dashed line) and von Neumann entropy S (red solid line); (b) probabilities p_0 and p_{\pm} of finding the ground state in appropriate two-site product states (as explained in the main text); and (c) fidelity \mathcal{F}_S (\mathcal{F}_T) between the ground state of the system and the singlet (triplet) qutrit–qutrit Bell state.

the probability corresponding to other states is present in the system). Such a switching can be realized in practical realizations, for instance, by adiabatic changes of the parameters of the Hamiltonian. Moreover, in Fig. 3(c) we can see that the fidelity \mathcal{F}_S increases and becomes larger than \mathcal{F}_T for $\theta > \theta_0$ and then reaches its maximal value ($\mathcal{F}_S = 1$) for vanishing biquadratic interaction ($\theta = 0$). At this point, the singlet Bell state is generated,

$$|G\rangle \stackrel{\theta=0}{=} |3, B_S\rangle, \quad (13)$$

which is manifested by $S = \log_2 3$ and $\mathcal{K} = 3$.

For attractive interactions ($\theta > 0$), the ground state remains almost exactly in the singlet state $|3, B_S\rangle$, i.e., fidelity \mathcal{F}_S decreases but not more than $\sim 10\%$ from unity. Moreover, we see that for this case entropy decreases at the same rate as the number of relevant orbitals. In the limit $\theta \rightarrow \pi/2$, we have $\mathcal{K} = 2$. This fact shows that only two relevant orbitals are involved, and might suggest that we are dealing with the same situation as that for $\theta = \theta_0$. However, for this situation entropy

$\log_2 3 > S > \log_2 2$. As a consequence, although the state of our system is defined in $3 \otimes 3$ Hilbert space, it is not a MES. The fact that $\mathcal{K} = 2$ is caused by the distribution of the probabilities p_0 and p_{\pm} . As we can see in Fig. 3(b), for such a situation the probability p_0 plays a dominant role and is four times higher than p_{\pm} . Therefore, the sum of both the probability amplitudes, corresponding to p_{\pm} , is equal to the amplitude corresponding to p_0 . As a consequence, the effective number of relevant orbitals is equal to 2.

B. Negative Linear Coupling ($\lambda = -1$)

Similarly, correlations in the ground state can be discussed for negative linear coupling $\lambda = -1$. As previously, for phases characterized by total spin $S^z = 2$ and $S^z = 1$, the ground state is the product state $|1, 1\rangle$ and the singlet qubit–qubit Bell state $||1; -\rangle\rangle$, respectively.

The situation changes for the case when total spin $S^z = 0$, which can be observed only for negative values of angle θ . For such a situation, all properties of the ground state can also be derived analytically. In particular, the formulas for entropy S and the number of relevant orbitals \mathcal{K} have the same form as

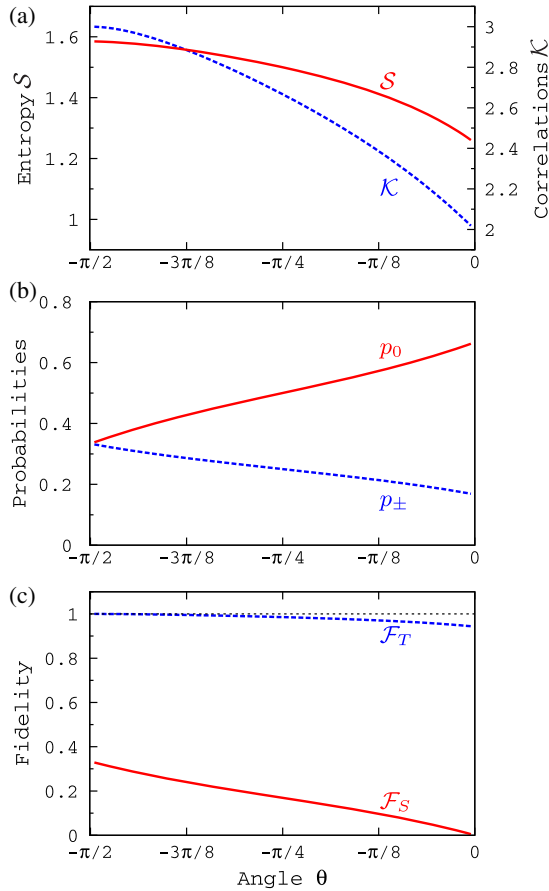


Fig. 4. Ground-state properties for negative linear coupling λ in the phase with total spin $S^z = 0$ as a function of mixing angle θ . (a) Number of relevant orbitals \mathcal{K} (blue dashed line) and von Neumann entropy S (red solid line); (b) probabilities p_0 and p_{\pm} of finding the ground state in appropriate two-site product states (as explained in the main text); and (c) fidelities \mathcal{F}_S and \mathcal{F}_T between the ground state of the system and the singlet and triplet qutrit–qutrit Bell states, respectively. In contrast to the case $\lambda = 1$, the ground state remains almost perfectly in the triplet qutrit–qutrit Bell state $|3, B_T\rangle$ in a whole phase.

in Eq. (10). They are plotted in Fig. 4(a). As can be seen in Fig. 4(b), in contrast to the case $\lambda = 1$, for negative linear coupling the ground state of the system remains almost perfectly in the triplet qutrit–qutrit Bell state $|3, B_T\rangle$. In the limit of infinite repulsive biquadratic interactions, the fidelity \mathcal{F}_T is equal to 1. Moreover, all three probabilities p_{\pm} and p_0 are equal to $1/3$. All these means that for cases when $\lambda = -1$, the degree of bipartite entanglement for the ground state is highly insensitive to the values of the parameters describing the superlattice. One should remember that such robustness appears when the system is in the phase with the vanishing total spin $S^z = 0$, and is not present for $\lambda = 1$ (see the cases discussed in the previous section).

6. CONCLUSIONS

We have discussed a model of two spin-1 bosons confined in the double-well potential of an optical superlattice and influenced by an external magnetic field. We have shown that depending on the values of the parameters appearing in the Hamiltonian, the ground state can belong to different phases distinguished by total spin S^z , and the presence of nonlinear biquadratic interaction considerably influences the system's properties. For the experimentally accessible values of the parameters, the ferro-, ferri-, and antiferromagnetic-like phases have been identified and presented as separate sectors in the phase diagram. We have pointed out that the sharp boundaries between the sectors are related to the crossings of energies in the spectrum of the Hamiltonian. Subsequently, we have uniquely determined the ground state of the system of a given magnetization for each magnetic phase. In this paper, we have concentrated on the possibility of the generation of a MES defined in $2 \otimes 2$ and $3 \otimes 3$ Hilbert spaces, and we have identified the set of parameters for which MESs of different kinds can be achieved. What is important is that by applying changes in the parameters describing the optical lattice, we can switch the system from one MES to another. For instance, we can continuously transform the state of our system from a singlet to a triplet Bell state. Moreover, it is possible to change the system's state from a MES defined in $2 \otimes 2$ Hilbert subspace to that is defined in $3 \otimes 3$ -dimensional space, and vice versa. Such a switching performed within a single phase can be induced by adiabatic changes of the parameters of the Hamiltonian. For cases where transitions are made between two phases, some additional interaction should be involved. Such interaction (for instance, interaction with an external bath) is necessary, as the system has to change its total spin, which commutes with the Hamiltonian.

In particular, we have found that in the ferrimagnetic-like phase the ground state is a qubit–qubit singlet Bell state regardless of the type of the Heisenberg interaction. Moreover, discussing the case of the antiferromagnetic-like phase, we have shown that the biquadratic term of Heisenberg interactions plays a crucial role in determining the properties of the ground state of the system. For $\lambda = -1$, the ground state is the qutrit–qutrit triplet Bell state, whereas for $\lambda = 1$ continuous transition from the qutrit–qutrit singlet to a qutrit–qutrit triplet can be induced by adiabatically varying the biquadratic interaction. Moreover, at the transition point, the ground state becomes the qubit–qubit triplet Bell state.

We believe that the system studied here can be a potential candidate for practical realization of a device that could be

applied as a switchable tool for generation of various MESs on demand.

ACKNOWLEDGMENTS

We would like to thank Prof. A. Miranowicz for his valuable suggestions and discussion. The authors wish to thank the (Polish) National Science Center (grant no. DEC-2011/01/D/ST2/02019) for support. Numerical calculations were performed in WCSS, Wrocław (Poland).

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