Dynamics and decoherence of two cold bosons in a one-dimensional harmonic trap

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We study dynamics of two interacting ultracold Bose atoms in a harmonic oscillator potential in one spatial dimension. Making use of the exact solution of the eigenvalue problem of a particle in the δ-like potential, we study the time evolution of an initially separable state of two particles. The corresponding time-dependent single-particle density matrix is obtained and diagonalized, and single-particle orbitals are found. This allows us to study decoherence as well as creation of entanglement during the dynamics. The evolution of the orbital corresponding to the largest eigenvalue is then compared to the evolution according to the Gross-Pitaevskii equation. We show that if initially the center of mass and relative degrees of freedom are entangled, then the Gross-Pitaevskii equation fails to reproduce the exact dynamics and entanglement is produced dynamically. We stress that predictions of our study can be verified experimentally in an optical lattice in the low-tunneling limit.

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I. INTRODUCTION

Theoretical description of a Bose-Einstein condensate of trapped weakly interacting atomic system is traditionally based on a mean-field approximation [1]. By assuming that many-body wave function can be written in a form of N-fold product state, that is, that all atoms occupy the same single-particle orbital, the stationary Gross-Pitaevskii (GP) equation for the order parameter is found. Assuming further that the N-fold product approximation holds also in dynamical situations, one arrives at the time-dependent Gross-Pitaevskii equation. Under most of experimental conditions, there are no strong correlations in the system, and the GP equation turned out to be extremely useful in predicting and describing a variety of features of those systems. Also, high-energy solutions of the GP equation can be useful in studying Bose systems at finite temperatures. The GP equation has become a workhorse theory of weakly interacting ultracold bosons.

However, examples when the mean-field description does not reproduce the real dynamics have been studied. For instance, direct comparison of the mean-field and many-body theories of vortex nucleation showed that the GP equation fails to describe this phenomenon [2–4]. Similarly, a mean-field description of attractive Bose systems encounters difficulties [5–7]. Due to permanent progress in experimental techniques, the physics of ultracold atomic gases started to penetrate areas traditionally associated with condensed-matter physics where correlations play a crucial role. Evidently, in such situations, a simple mean-field description based on the GP equation becomes insufficient. The Mott insulator-superfluid transition [8] or the Tonks-Girardeau gas [9,10] are some examples.

It is commonly believed that creation of the Mott insulator with a small and controlled number of atoms per lattice site allows for applications of such systems in quantum information. All quantum information processing is based on the quantum correlations, which cannot be described by any classical theory based on local realism. States showing these nonlocal correlations are known as entangled states. Two entangled spin-$\frac{1}{2}$ particles are the essence of the Einstein, Podolsky, and Rosen paradox [11]. The entangled states violate Bell inequalities [12]. It has been realized that also continuous-variable entangled systems can be used in quantum computations [13–19]. Several authors [20–22] studied recently creation of entanglement during dynamics of two interacting particles in a harmonic trap. In particular, it has been shown that this dynamically entangled state violates a Bell-type inequality for a certain choice of observables [20].

In this article, we study dynamically created entanglement and its measure (the von Neuman entropy) for a realistic system of two identical bosonic atoms in a harmonic trap. We consider low-energy collisions of the atoms. At such energies, the range of van der Waals interactions is smaller than the s-wave scattering length. Therefore, the interaction potential can be approximated by a contact pseudopotential. This approximation is in excellent agreement with experimental results [23] where the binding energy of molecular system has been measured. The molecules were created from atoms in an optical lattice in the limit of small tunneling. This experimental arrangement is perfectly suited for a study of exact dynamics of two trapped atoms. We consider a realistic case of two atoms per lattice site deep in the Mott insulator phase. By applying Bragg pulses [24], one creates a state in which each atom is in a superposition of two counterpropagating wave packets. Initially, such a state is a two-fold product state of two identical wave functions; that is, it is separable. Each wave function has two components moving initially with opposite momenta. The center of mass dynamics is generated by a different Hamiltonian than dynamics of the relative coordinate; therefore, this two-particle continuous-variable system becomes entangled. We use the von Neuman entropy as a measure of the entanglement and study its behavior in time for various interaction strengths. We also analyze a coherence of the system which is directly related to the largest eigenvalue of the single-particle density matrix and compare the exact dynamics to the mean-field description based on the GP equation.

II. TWO BOSONS IN A HARMONIC TRAP

We are going to study dynamics of the simplest nontrivial system—two atoms confined in a one-dimensional harmonic potential. In fact, generalization of our results to two or three spatial dimensions is straightforward. We limit our analysis to the one-dimensional (1D) case as this situation captures all features of the dynamics. For simplicity we are using harmonic-oscillator units. It means that all energies are measured in $\hbar\omega$, all lengths are measured in $\sqrt{\hbar/m\omega}$, and all momenta are measured in $\sqrt{\hbar m\omega}$. The Hamiltonian of the system of two interacting bosons in the harmonic trap has the form

$$\mathcal{H} = -\frac{1}{2}\frac{\partial^2}{\partial x_1^2} - \frac{1}{2}\frac{\partial^2}{\partial x_2^2} + \frac{1}{2}\left(x_1^2 + x_2^2\right) + g\delta(x_1 - x_2), \tag{1}$$

where x_1 and x_2 are positions of atoms interacting via a short-range potential modeled by the δ function. This form of the short-range interaction is justified in the limit of vanishing relative velocity of colliding atoms, where atomic de Broglie wavelength is much larger then a range of two-body potential. In two and three dimensions, the corresponding Hamiltonian is not a self-adjoint operator. To correct for this fact, a regularization is required. In contrast to many dimensions, the regularization of the δ function is not necessary in the 1D case [25]. In one dimension, the parameter g is given by $g = -2/a_0$, where a_0 is a scattering length [26]. It is worth noting that finite-range interactions between particles modeled by the Gaussian function in the context of the dynamics of two bosons were studied in [27].

To demonstrate entanglement formation, we study the evolution of two bosons which initially are in a product quantum state:

$$\Psi_0(x_1,x_2) = \Phi_0(x_1)\Phi_0(x_2). \tag{2}$$

Function $\Phi_0(x)$ is a one-particle wave function called the order parameter in the mean-field context.

The exact dynamics of the two interacting bosons in the harmonic trap can be found because all eigenstates of the full two-body Hamiltonian (1) are known. They are found in [28]. The two-particle problem has to be first brought to a single-particle one by introducing the center of mass and the relative coordinates:

$$X = \frac{1}{\sqrt{2}}(x_1 + x_2), \tag{3a}$$

$$\xi = \frac{1}{\sqrt{2}}(x_1 - x_2). \tag{3b}$$

In these coordinates, Hamiltonian (1) separates into two independent parts—the center of mass part and the relative part:

$$\mathcal{H}_{\text{c.m.}} = -\frac{1}{2}\frac{d^2}{dX^2} + \frac{1}{2}X^2, \tag{4a}$$

$$\mathcal{H}_{\text{rel}} = -\frac{1}{2}\frac{d^2}{d\xi^2} + \frac{1}{2}\xi^2 + \frac{\sqrt{2}}{2}g\,\delta(\xi). \tag{4b}$$

As one can see, the dynamics of the center of mass is described by the standard 1D harmonic oscillator Hamiltonian (4a). Its eigenstates are well known and have a standard form

$$\chi_n(X) = \frac{\pi^{-1/4}}{\sqrt{2^n\,n!}}\,H_n(X)\,e^{-X^2/2}, \tag{5a}$$

where $\mathrm{H}_n(x)$ are Hermite polynomials. The energy of the nth eigenstate in our units is obviously given by

$$\mathcal{E}_n = n + \tfrac{1}{2}. \tag{5b}$$

The eigenstates of the Hamiltonian (4b) describing the relative dynamics of two particles are given in [28] and for the 1D problem have a form

$$\varphi_m(\xi) = \frac{\pi^{-1/4}}{\sqrt{2^m\,m!}}\,H_m(\xi)\,e^{-\xi^2/2}, \quad m \text{ odd}, \tag{6a}$$

$$\varphi_m(\xi) = \mathcal{N}_m\,U\left(-\nu_m, \frac{1}{2}, \xi^2\right)\,e^{-\xi^2/2}, \quad m \text{ even}, \tag{6b}$$

where $U(\alpha,\beta,x)$ are confluent hypergeometric functions and $\mathcal{N}_m$ are normalization coefficients. Since the wave function of identical bosons must be symmetric under exchange of the two particles, the physical wave function is composed from functions with even m only. The energies E_m of these even states are given by a sequence of zeros of the function:

$$f(E) = \frac{\Gamma(-E/2 + 3/4)}{\Gamma(-E/2 + 1/4)} - \frac{1}{a_0}. \tag{7}$$

The quantum number ν is equal to $\nu_m = (2E_m - 1)/4$. The initial wave function can be easily decomposed to the superposition of the eigenstates of the Hamiltonian:

$$\Psi_0(\xi,X) = \sum_{nm} \alpha_{nm}\,\chi_n(X)\varphi_m(\xi). \tag{8}$$

Obviously, the evolution of the initial two-boson state is given by

$$\Psi(\xi,X,t) = \sum_{nm} \alpha_{nm}\,\chi_n(X)\varphi_m(\xi)\,e^{-i(\mathcal{E}_n + E_m)t}. \tag{9}$$

The last step is to return to the original coordinates by using relations (3):

$$\Psi(x_1,x_2,t) = \sum_{nm} \alpha_{nm}\,\chi_n\left(\frac{x_1 + x_2}{\sqrt{2}}\right)\varphi_m\left(\frac{x_1 - x_2}{\sqrt{2}}\right)e^{-i(\mathcal{E}_n + E_m)t}. \tag{10}$$

Standard methods of detection of ultracold trapped atomic systems are destructive. The optical lattice potential is turned off, and the system is allowed to expand ballistically. Only after expansion does the size of the system exceed the resolution of a charge-coupled device (CCD) camera. The picture of the CCD camera gives therefore direct insight into the initial momentum distribution of atoms. The wave function of Eq. (10) written in the momentum space of the two atoms is

$$\psi(k_1,k_2,t) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 e^{-ik_1x_1} e^{-ik_2x_2}\,\Psi(x_1,x_2,t). \tag{11}$$

In repeated single-particle detections preceded by the ballistic expansion of the system, one-particle momentum distribution is monitored:

$$n_{\texttt{Exact}}(k,t) = \rho_1(k,k,t), \tag{12}$$

where $\rho_1(k,k',t)$ is the reduced one-particle density matrix in the momentum representation:

$$\rho_1(k,k',t) = \int_{-\infty}^{\infty} dk_2\, \psi^*(k,k_2,t)\psi(k',k_2,t). \qquad (13)$$

By making its spectral decomposition, we can determine the number of orbitals and their relative occupations needed for accurate description of the two-boson dynamics. Time dependence of the eigenvalues of the density matrix is discussed later. Let us mention that the largest eigenvalue is a direct measure of the coherence of the system.

We compare this exact dynamics with the approximate one governed by the GP equation. The main idea leading to the mean-field approximation relies on the assumption that generation of entanglement between bosons during the evolution is negligible and therefore the quantum state of the system remains separable. In other words, all correlations between bosons are neglected and the same wave functions of every particle are assumed during the evolution:

$$\Psi(x_1,x_2,t) = \Phi(x_1,t)\Phi(x_2,t). \qquad (14)$$

This assumption leads directly to the GP equation, which determines the dynamics of the one-particle wave function $\Phi(x,t)$:

$$i\partial_t\Phi(x,t) = \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^2 + g|\Phi(x,t)|^2\right)\Phi(x,t). \qquad (15)$$

The probability density in momentum space reads

$$n_{\mathrm{GP}}(k,t) = |\phi(k,t)|^2, \qquad (16)$$

where $\phi(k,t)$ is the Fourier transform of the time-dependent solution of the GP equation, $\phi(k,t) = \int dx\, e^{-ikx}\Phi(x,t)$. We compare the exact one-particle momentum distribution (12) with that predicted by the GP approximation (16). Moreover, in the situation when many eigenvalues of the density matrix (13) are of the same order, we can also compare the GP momentum distribution (16) with the momentum distribution of the dominant orbital obtained from diagonalization of the exact one-particle density matrix in the momentum space. Obviously, in a general case the GP solution overestimates the coherence of the system.

The GP equation is solved numerically on a spatial grid of $N_p = 2^{10}$ points separated by $\delta x = 5 \times 10^{-2}$. The time step is equal to $\delta t = \pi/4 \times 10^{-3}$. We use the split-operator method, which is very stable for the chosen temporal and spatial steps.

III. RESULTS

To make the detailed comparison, we concentrate on one particular class of the initial states. We assume that at the beginning each particle is in the state described by the Schrödinger cat–like wave function

$$\Phi_0(x) = \mathcal{N}\left[e^{-(x-L)^2/2} + e^{-(x+L)^2/2}\right]. \qquad (17)$$

Parameter L measures the separation between two wave packets moving in the opposite direction in the relative coordinates space. Such a choice is motivated by the preparation procedure described previously: we assume that Bragg pulses bringing the atoms into the superposition of wave packets moving in opposite directions are applied. When $L = 0$, then the initial state is very close to the ground state of the system, so we expect that the exact dynamics is almost indistinguishable from the dynamics in the mean-field approximation. For large L, the initial state is still separable but highly delocalized. Relative and center of mass degrees of freedom are entangled in the initial state. They evolve in a different way; therefore, we expect that the exact dynamics could be dramatically different than the dynamics predicted by a simple mean-field approach.

A. Dependence on delocalization of one-particle state

First let us discuss the situation for generic interaction strength $g = -0.2$ ($a = 10$) when $L = 1$, that is, when the extension of the initial state is equal to the trap length unit. We observe that the single-particle density matrix obtained from the exact dynamics develops more then one nonzero eigenvalue, that is, many one-particle orbitals are involved. Figure 1(a) shows time dependence of the eigenvalues of the one-particle density matrix (13). Because one of the eigenvalues is incessantly much larger than the others, the

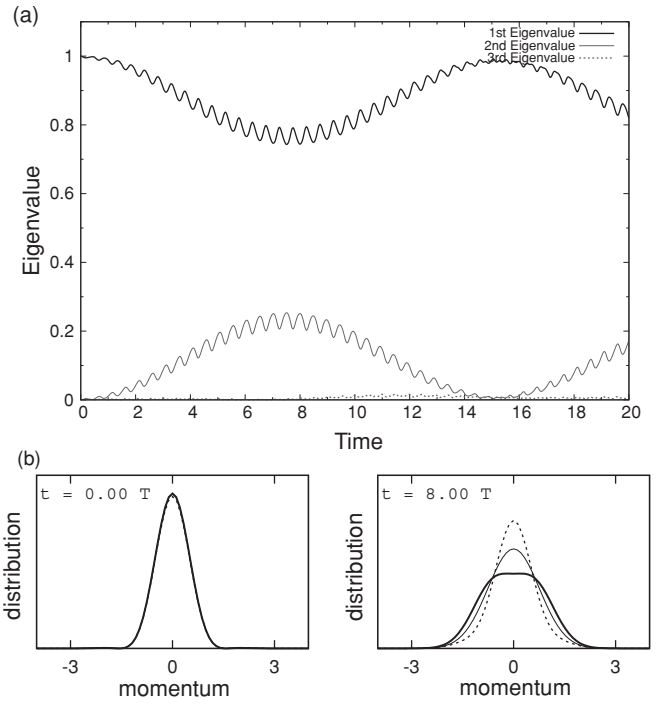

FIG. 1. (a) Eigenvalues of the one-particle density matrix (13). Unit of time is equal to the period of the trap. In this situation (parameters: $g = -0.2$, $L = 1$), the initial state is not far from the ground state of the system. One eigenvalue still dominates, and therefore the system should be well described by the mean-field approximation. (b) Two plots present the one-particle momentum distributions (in dimensionless units) predicted by the exact (thick solid line) and the GP solutions (dotted line) in two interesting moments. The third (thin solid) line comes from the exact solution and presents the momentum distribution of the first orbital. As was expected, all three predictions are almost the same for the considered set of parameters. A movie presenting the time evolution of momentum distributions is available online [29].

system coherence is large and the GP description is quite accurate in this case. Time dependence of the momentum distributions deduced from the GP equation and the exact solution are shown in Fig. 1(b), and they are in agreement with our predictions (whole time dependence of momentum distributions is available online [29]).

The situation changes dramatically when we increase the delocalization parameter. When L is large enough, then a few orbitals can play the crucial role in the dynamics and the mean-field approximation is no longer valid. Figure 2(a) shows the time dependence of the eigenvalues of the density matrix for $L = 3$. As we see, the main orbital (its eigenvalue is represented by a thick solid line) initially dominates, but after a few periods of the trap oscillations, the other orbital becomes much more important than the first one. The dynamics is obviously much more complicated than predicted by the mean-field approach. It is clear when we compare the momentum density distribution predicted by the exact and the mean-field solutions [Fig. 2(b) and movie available online [29]].

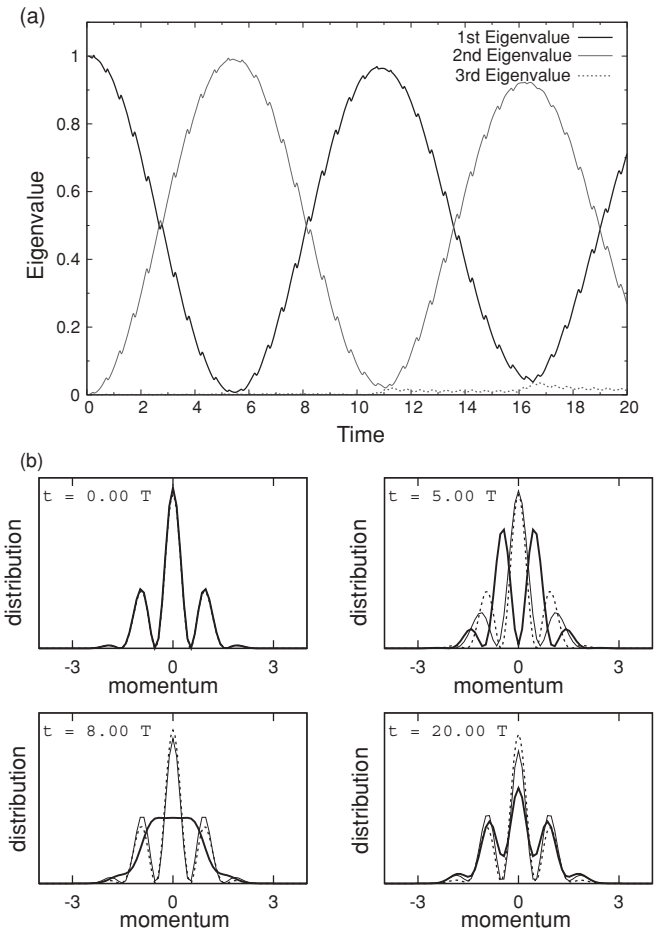

FIG. 2. (a) Eigenvalues of the one-particle density matrix (13) for $g = -0.2$, $L = 3$. Unit of time is equal to the period of the trap. In this situation, the initial state is a product of highly delocalized one-particle wave functions. There is no one dominant eigenvalue during the evolution, and therefore the GP equation will not predict dynamics correctly. (b) Time dependence of the one-particle momentum distribution (in dimensionless units) predicted by the exact (thick solid line) and GP solutions (doted line). As long as the first eigenvalue dominates during the time evolution, the predictions are almost the same. After five periods, the second eigenvalue is the largest one and therefore the predictions are highly different. Solutions of the exact and GP dynamics become similar after eleven trap periods, when the first eigenvalue starts to dominate again. Notice that third (thin solid) line presenting momentum distribution of first one-particle orbital of an exact solution recovers predictions of the GP equation. The movie is available on-line [29].

We see that evidently the GP equation properly describes the dynamics of the first orbital rather then the whole system [Fig. 2(b)]. It is the reason why the GP equation gives good predictions when only one eigenvalue of the one-particle matrix dominates during the entire evolution.

It is also interesting to study similarities and differences between prediction of the mean-field approach and the exact solution in the situation when the initial state of each particle is antisymmetric in position space, that is, described by the wave function of the form

$$\Phi_0(x) = \mathcal{N}\left[e^{-(x-L)^2/2} - e^{-(x+L)^2/2}\right]. \tag{18}$$

Nevertheless, the wave function of the system is still symmetric under particle exchange. Since the corresponding GP Hamiltonian is invariant under reflection $x \to -x$, the symmetry of the initial state will be preserved. As we observe, it is not true for the exact two-body dynamics. The evolution preserves only the symmetry of each orbital separately but not the symmetry of the whole system. It is clearly demonstrated in Fig. 3 where we compare the momentum distribution predicted by the mean-field approach with the single-particle density obtained from

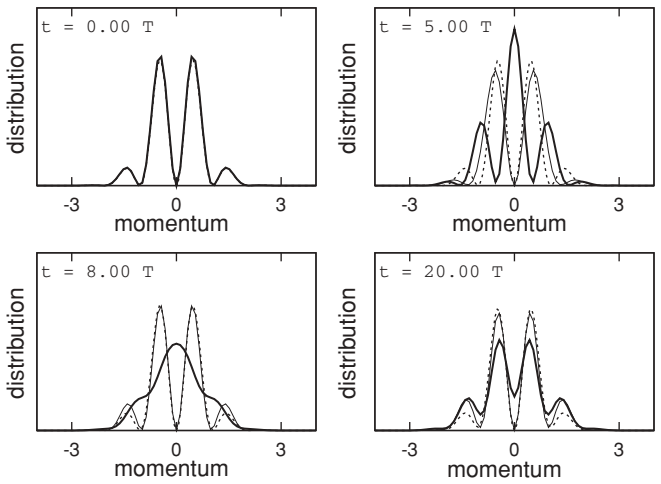

FIG. 3. Time dependence of the one-particle momentum distribution (in dimensionless units) for the antisymmetric initial state $\Phi_0(x) = \mathcal{N}[e^{-(x-L)^2/2} - e^{-(x+L)^2/2}]$ with $L = 3$. Thick solid line represents the density predicted by an exact solution, while the dotted one represents the density coming from the mean-field approach. Properties of the GP equation provide that if the function $\Phi_0(x)$ is antisymmetric under $x \to -x$ symmetry, then it will stay antisymmetric during the whole evolution. It is not true for the one-particle density predicted by an exact solution. The thin solid line presents momentum distribution of the first one-particle orbital of the exact solution. As we see, its spatial reversal symmetry is preserved during the evolution. It shows once more that the GP equation describes properly the dynamics of the first orbital only. A whole movie is available online [29].

the exact dynamics. Time dependence of the eigenvalues of one-particle density matrix is identical to the one for the corresponding symmetric case [Fig. 2(a)]. A movie is also available online [29].

B. Dependence on interaction strength

Now we show that correctness of the mean-field approximation significantly depends on the interaction strength parameter g. It is quite obvious that in the situation when the interaction is switched off, the two particles initially in the state which is not entangled (product state) will stay in such a state during the whole evolution even for a highly delocalized state. In this case, the mean-field approximation naturally leads to the same solution as the exact solution. It is the interparticle interaction which can produce entangled two-body states during the evolution.

Time dependence of the eigenvalues for a moderate interaction strength ($g = -0.2$) is presented in Figs. 1 and 2. In those situations, only two eigenvalues (i.e., two orbitals) are important for many trap periods. For stronger interactions, this picture changes significantly. Time evolution of eigenvalues for strong interaction ($g = -0.4$) and $L = 2$ are shown in Fig. 4. After a few trap periods, many different orbitals become important. Moreover, the orbital which dominates at the beginning becomes unimportant after a very short time. Therefore, we do not expect that the GP approximation may give correct predictions in this case.

On the other hand, when the interaction is very weak, we can expect that the production of entanglement will be very slow even for highly delocalized states and therefore the mean-field approximation may be correct for a long evolution time. Time dependence of the eigenvalues of the one-particle density matrix when the interaction is weak but the initial state is highly delocalized is presented in Fig. 5.

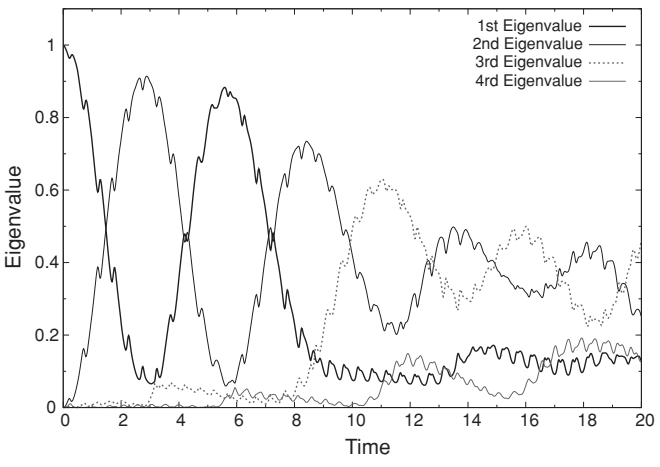

FIG. 4. Eigenvalues of the one-particle density matrix as functions of time for parameters $g = -0.4$, $L = 2$. The unit of time is equal to the period of the trap. The interaction between bosons is strong, and the initial state of one particle is highly delocalized. In such a situation, many orbitals play a crucial role during the evolution of the system. Therefore, the exact dynamics is much more complicated that the dynamics predicted by the mean-field approximation.

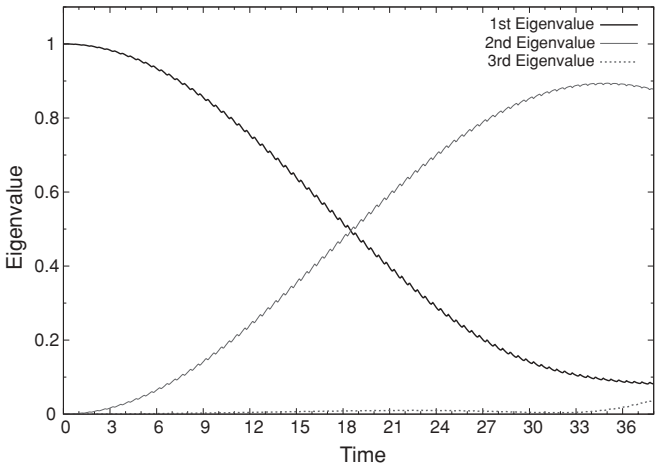

FIG. 5. Eigenvalues of the one-particle density matrix as functions of time for parameters $g = -0.04$, $L = 2$. Unit of time is equal to the period of the trap. In this situation, the interaction between bosons is very weak, but the initial state is far from the ground state of the system. During the first eighteen trap periods, only one eigenvalue dominates, and therefore the dynamics of the system can be quite correctly described by the mean-field approximation for a long time. Notice that the time scale is two times larger than in the previous situation.

C. Revivals of product states

By looking at Figs. 2 and 5, one can observe that as initially only one eigenvalue dominates in the Schmidt decomposition of the single-particle density matrix, the other eigenvalues become more important at later times. However, the time dependence of the dominant eigenvalue exhibits some oscillations, and a partial revival of the "initial" eigenvalue can be observed. It is interesting to find a physical explanation of this behavior. To this end, in Fig. 6 we show the spectrum of the two-particle state for the two studied parameter sets $g = -0.2$ and $L = 3$ (as in Fig. 2) and $g = -0.4$ and $L = 2$

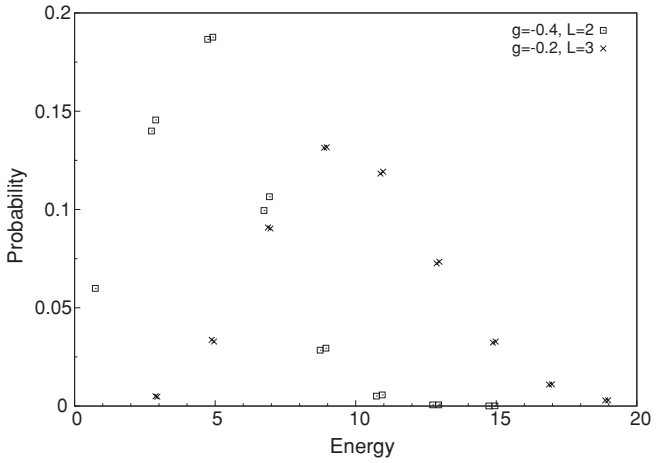

FIG. 6. Energy spectrum of two initial states described by the parameters $g = -0.4$ and $L = 2$ (squares) and $g = -0.2$ and $L = 3$ (crosses). Note that both spectra are well picked and energies are paired. Unit of energy is equal to the characteristic energy of the harmonic trap $\hbar\omega$.

(as in Fig. 4). In this figure, we plot the probability of the given eigenenergy, $|\alpha_{nm}|^2$, resulting from the decomposition (8).

First, let us notice that the eigenenergies do appear in pairs. The effect of pairing of eigenenergies can be easily explained. For given n and m, the eigenenergy has two components. $\mathcal{E}_n$ is the energy of the center of mass while E_m corresponds to the relative coordinate. As mentioned before, energies of the relative excitations are very close to the energies of harmonic trap. This is because the potential in the relative coordinate space is the harmonic potential of the trap modified at $\xi = 0$ by the presence of δ function. This δ function shifts the harmonic energy by a very small amount. Therefore, the state labeled by (n,m) is almost degenerate with the state labeled by (m,n), and the spectrum is paired.

If by Δ we denote the difference between two energies of the dominant pair (maximum in the spectral decomposition), it is clear that after time $T_R = 2\pi/\Delta$ these two, the most important amplitudes of the two-atom wave function, match in phase and the partial revival of the product state can be observed. This is signified by a reappearance of the initial eigenvalue of the single-particle matrix. The revival time calculated this way for parameters of Fig. 2 is equal to $T_R = 10.96$, while for Fig. 4 it is $T_R = 5.56$, which agrees perfectly with predictions of the exact dynamics.

Obviously revival time T_R depends on the initial state as well as eigenmodes of the Hamiltonian (1).

In addition to the large time oscillations of the eigenvalues of the density matrix, some small fast oscillations can be also observed. These fast modulations appear every half of the trap period when two wave packets meet at the trap center and result from interaction between them.

D. Entanglement of particles

Mutual interactions between particles obviously leads to the quantum correlations between particles. To study them, we use the correlation measure introduced in [30]:

$$\mathbf{K}(\rho_1) = \left(\sum_i \lambda_i^2\right)^{-1}, \tag{19}$$

where λ_i are the eigenvalues of the one-particle density matrix ρ_1. This measure has very simple interpretation. It gives an effective number of single-particle orbitals occupied in the given many-body state. In particular, when the one-particle density matrix has n equal eigenvalues, then $\mathbf{K} = n$.

One other commonly used [20–22,31] measure of entanglement in the system is von Neumann entropy defined as

$$\mathbf{S}(\rho_1) = -\mathrm{Tr}\,(\rho_1 \ln \rho_1) = -\sum_i \lambda_i \ln \lambda_i. \tag{20}$$

This entropy is even more interesting than the number of dominant eigenvalues $\mathbf{K}$ since it is directly connected with the entropy defined in the thermodynamical context. Time dependence of these two measures of entanglement in the system for two different regimes of interaction strength are presented in Fig. 7. Obviously in the beginning, when the system is in a separable state, entanglement and von Neumann entropy are equal to 1 and 0 respectively. We observe that correlation $\mathbf{K}$ and entropy $\mathbf{S}$ increase in time and seem

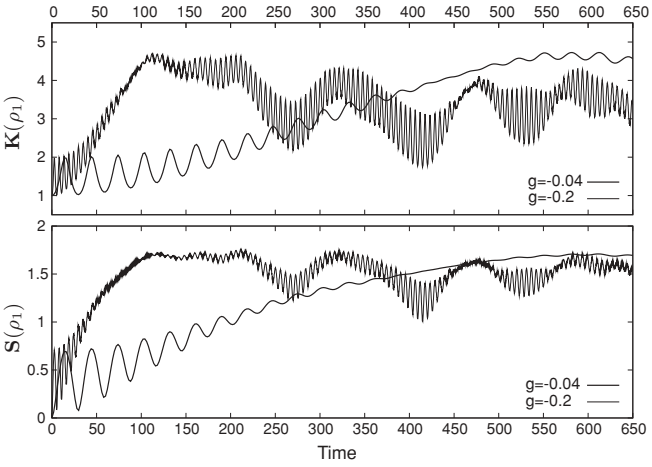

FIG. 7. Time dependence of the number of dominant eigenvalues $\mathbf{K}$ defined in (19) and of the von Neumann entropy $\mathbf{S}$ defined in (20) for $g = -0.04$ (thick line) and $g = -0.2$ (thin line). Other parameters are the same as in Fig. 5. Unit of time is equal to the period of the trap.

to saturate for large time. Even though they have different physical interpretations, they behave very similarly, which might seem quite surprising. They reach a stationary regime faster for stronger interactions.

Both quantities exhibit fast oscillations modulated by slowly varying functions. These fast oscillations are related to partial revival of dominant eigenvalue discussed in the previous subsection. Every minimum observed in correlation function corresponds to the moment when there is a dominant eigenvalue in the Schmidt decomposition of the one-particle density matrix. Let us remember that these revivals are related to phase matching of two dominant eigenmodes of the two-particle state.

Long time modulations of correlation functions are related to the quantum nature of the system and discreetness of the energy spectrum. In such cases, evolution is always quasiperiodic and due to the interference of amplitudes long time-scale oscillations do appear. In our case, the number of modes with no zero amplitudes is relatively small, and therefore oscillations of correlation functions appear on a time scale of a few hundred trap periods.

IV. SUMMARY

In this article, we study the exact dynamics of two particles trapped in a harmonic trap and interacting by a contact potential. We assumed that initially each particle is transferred by the Bragg pulses to the state where the superposition of two wave packets moves in opposite directions. We show that the two-particle state, although initially being a product state, does not preserve the product form during the evolution. The reason is that the initial state entangles the center of mass and relative coordinates of the two-particle system. These two degrees of freedom evolve according to different Hamiltonians. As a result, the single-particle reduced matrix develops many eigenvalues during the evolution that signify the decreasing coherence of the system. This situation cannot be correctly described by the GP equation. Our predictions can be verified in the experiment with deep optical lattices when

two atoms occupy each site. We show one-particle momenta distributions for different initial states and compare them to those obtained from the mean-field dynamics. The differences between the two signify the two-atom entanglement. The momentum distribution is directly measured by exposure of the system to a resonant light after ballistic expansion and therefore creation of entanglement in the two-particle system can be easily traced in time and compared to exact solutions. We monitored the von Neumann entropy, which is a common measure of entanglement. We show that entanglement is dynamically created during evolution; however, it is not very surprising for an interacting system. A comment about a system of two fermionic particles would be also appropriate. As two identical fermions do not interact in the s-wave channel, as long as other partial waves can be neglected, their dynamic is driven by the noninteracting Hamiltonian. On the other hand, if the spatial part of a wave function of two fermions is symmetric and a spin part is responsible for the antisymmetrization of the total wave function, then our exact solution evidently applies to such a situation.

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[1] F. Dalfovo, S. Giorgini, L. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[2] D. Dagnino, N. Barberán, M. Lewenstein, and J. Dalibard, Nature Physics **5**, 431 (2009).
[3] D. Dagnino, N. Barberán, and M. Lewenstein, Phys. Rev. A **80**, 053611 (2009).
[4] B. Juliá-Díaz, D. Dagnino, M. Lewenstein, J. Martorell, and A. Polls, Phys. Rev. A **81**, 023615 (2010).
[5] M. Gajda, Phys. Rev. A **73**, 023603 (2006).
[6] M. Gajda, M. Zaluska-Kotur, and J. Mostowski, Opt. Express **8**, 106 (2001).
[7] O. E. Alon, Alexej I. Streltsov, and Lorenz S. Cederbaum, Phys. Rev. A **77**, 033613 (2008).
[8] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, Nature **415**, 39 (2002).
[9] B. Paredes, A. Widera, V. Murg, O. Mandel, S. Fölling, I. Cirac, G. V. Shlyapnikov, T. W. Hänsch, and I. Bloch, Nature **429**, 277 (2004).
[10] T. Kinoshita, T. Wenger, and D. S. Weiss, Science **305**, 1125 (2004).
[11] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[12] J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966).
[13] L. Vaidman, Phys. Rev. A **49**, 1473 (1994).
[14] S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. **80**, 869 (1998).
[15] P. van Loock, S. L. Braunstein, and H. J. Kimble, Phys. Rev. A **62**, 022309 (2000).
[16] P. van Loock and S. L. Braunstein, Phys. Rev. Lett. **84**, 3482 (2000).
[17] P. van Loock and S. L. Braunstein, Phys. Rev. Lett. **87**, 247901 (2001).
[18] S. L. Braunstein, Nature (London) **394**, 47 (1998).
[19] A. Furusawa, J. L. Sørensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik, Science **282**, 706 (1998).
[20] H. Mack and M. Freyberger, Phys. Rev. A **66**, 042113 (2002).
[21] B. Sun, D. L. Zhou, and L. You, Phys. Rev. A **73**, 012336 (2006).
[22] J. Goold, L. Heaney, T. Busch, and V. Vedral, Phys. Rev. A **80**, 022338 (2009).
[23] T. Stöferle, H. Moritz, K. G. Günter, M. Khöl, and T. Esslinger, Phys. Rev. Lett. **96**, 030401 (2006).
[24] M. Kozuma, L. Deng, E. W. Hagley, J. Wen, R. Lutwak, K. Helmerson, S. L. Rolston, and W. D. Phillips, Phys. Rev. Lett. **82**, 871 (1999).
[25] P. Šeba, Czech. J. Phys. **36**, 559 (1986).
[26] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics* (Springer-Verlag, New York, 1988).
[27] C. Matthies, S. Zöllner, H.-D. Meyer, and P. Schmelcher, Phys. Rev. A **76**, 023602 (2007).
[28] T. Busch, B. Englert, K. Rzążewski, and M. Wilkens, Found. Phys. **28**, 549 (1998).
[29] See supplementary material at [http://link.aps.org/supplemental/10.1103/PhysRevA.82.053631] for movies presenting time dependence of momentum distributions predicted by exact solution and in the mean-field approximation.
[30] R. Grobe, K. Rzążewski, and J. H. Eberly, J. Phys. B **27**, L503 (1994).
[31] R. Paškauskas and L. You, Phys. Rev. A **64**, 042310 (2001).