Anderson localization of electromagnetic waves in confined dielectric media

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Anderson localization of electromagnetic waves in random arrays of dielectric cylinders confined within a planar metallic waveguide is studied. The disordered dielectric medium is modeled by a system of randomly distributed two-dimensional electric dipoles. An effective theoretical approach based on the method of images is developed. A clear distinction between isolated localized waves (which exist already in finite media) and the band of localized waves (which appears only in the limit of the infinite medium) is presented. The Anderson localization emerging in the limit of an infinite medium is observed both in finite-size scaling analysis of transmission and in the properties of the spectra of some random matrices. [S1063-651X(99)04203-8]

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I. INTRODUCTION

The concept of Anderson localization originates from investigations of transport properties of electrons in noncrystalline systems such as amorphous semiconductors or disordered metals [1]. As pointed out by Anderson, in a sufficiently disordered solid an entire band of spatially localized electronic states can be formed [2,3]. It is natural to expect that the material in which an entire band of localized electronic states exists will be an insulator, whereas the case of extended states will correspond to a conductor [4]. In this way the phenomenon of Anderson localization may be referred to as a metal-insulator transition [5]. It is a common belief that localization is a pure interference effect and originates from multiple scattering of an electron from randomly distributed defects in the solid. On the other hand, the multiple scattering and interference are well known phenomena for electromagnetic waves and, consequently, several generalizations of electron localization to the realm of electromagnetic waves have been proposed [6–10].

The Anderson transition in the propagation of electromagnetic waves can best be observed experimentally in the scaling properties of the transmission $T$. Imagine a slab of thickness $L$ containing randomly distributed nonabsorptive scatterers. Usually propagation of electromagnetic waves in weakly scattering random media can be described adequately by a diffusion process [11,12]. Thus the equivalent of Ohm’s law holds and the transmission decreases linearly with the thickness of the slab, i.e., $T \propto L^{-1}$. However, when the fluctuations of the dielectric constant become large enough, the electromagnetic field ceases to diffuse and becomes localized due to interference. Anderson localization occurs when this happens. In such a case the material behaves as an optical equivalent of an insulator and the transmission decreases exponentially with the size of the system $T \propto e^{-L/\xi}$ [7,13].

A convincing experimental demonstration that Anderson transition is indeed possible in three-dimensional disordered dielectric structures has been given recently [14]. The strongly scattering medium has been provided by semiconductor powders with a very large refractive index. By decreasing the average particle size it was possible to observe a clear transition from linear scaling of transmission ($T \propto L^{-1}$) to an exponential decay ($T \propto e^{-L/\xi}$). Some localization effects have been also reported in previous experiments on microwave localization in copper tubes filled with metallic and dielectric spheres [15]. However, the latter experiments were plagued by large absorption, which makes the interpretation of the data quite complicated.

Another experiment on microwave localization has been performed in a two-dimensional medium [16]. The scattering chamber was set up as a collection of dielectric cylinders randomly placed between two parallel aluminum plates on half the sites of a square lattice. These authors attributed the observed sharp peaks of transmission to the existence of localized modes and measured the energy density of the electromagnetic field localized by their random structures.

In this paper we develop a simple yet reasonably realistic theoretical approach to Anderson localization of electromagnetic waves in two-dimensional dielectric media confined within a metallic waveguide. The results of our previous papers [17,18] dealing with the free space configurations are now extended to encompass the case of nontrivial boundary conditions. A sound physical interpretation in terms of transmission experiment is also proposed.

It should be stressed that the boundary conditions considered in this paper are different from those encountered in the experiment of Ref. [16]. To minimize the effect of the waves reflected off the edges of the scattering chamber, its perimeter was lined with a layer of microwave absorber. Therefore, to model that particular experiment, it is appropriate to use the free space boundary conditions (as we did in our previous papers [17,18]). Results presented here suggest, however, that it is easier to observe the localized states for dielectric cylinders confined to a metallic waveguide than in the case of open geometry. An experimental verification of our predictions seems feasible and it would allow a deeper understanding of localization phenomena in two-dimensional media.

By confining a system of randomly distributed dielectric cylinders into a planar metallic waveguide, we are able to observe clear signs of Anderson localization already for $N = 100$ scatterers. One of the indicators of localization is the phase transition in the spectra of certain random matrices.
already discovered (for free space situation) in our previous paper [18]. This property of random Green functions is now
generalized to the case of a confined dielectric medium
(where the Green function is different). It may be interpreted
as an appearance of the band of localized electromagnetic
waves emerging in the limit of the infinite medium. A con-
nection between this phenomenon and a dramatic inhibition
of the propagation of electromagnetic waves in a spatially
random dielectric medium is provided. A clear distinction
between isolated localized waves (which do exist in finite
confined media) and the band of localized waves (which ap-
pears in the limit of an infinite random medium) is also pre-

This paper is organized as follows. In Sec. II we introduce
the basic assumptions of our model. The explicit expressions
for the transmission and reflection coefficients are derived. In
Sec. III the method of images is recalled and applied for our
configuration. We arrive at the system of linear Lippmann-
Schwinger-like equations determining the polarization of the
medium for a given incident wave. In Sec. IV a numerical
transmission experiment is performed. Sharp peaks of trans-
mission are observed and attributed to the existence of local-
ized modes. The finite size scaling of transmission indicates
an exponential decay relevant for Anderson localization. In

II. BASIC ASSUMPTIONS

In the following we study the properties of the stationary
solutions of the Maxwell equations in two-dimensional me-
dia consisting of randomly placed parallel dielectric cylin-
ders of infinite height (i.e., very long as compared to the
wavelength of the electromagnetic field). This means that
one, say (y), out of three dimensions is translationally in-
variant and only the remaining two (x,z) are random. The
main advantage of this two-dimensional approximation is
that we can restrict ourselves to the scalar theory of electro-
magnetic waves [17]:

\[ \tilde{E}(\tilde{r},t) = \text{Re}\{ \tilde{e}r, \mathcal{E}(x,z) e^{-i\omega t} \}. \]  

Consequently, the polarization of the medium takes the form

\[ \tilde{P}(\tilde{r},t) = \text{Re}\{ \tilde{e}r, \mathcal{P}(x,z) e^{-i\omega t} \}. \]  

The model predictions can be compared with rigorous nu-
erical simulations [19–21].

Localization of electromagnetic waves in 2D media is usu-
ally studied experimentally in microstructures consisting
of dielectric cylinders with diameters and mutual distances
being comparable to the wavelength [16]. However, it seems
to be a reasonable assumption that what really counts for the
basic features of localization is the scattering cross section
and not the real geometrical size of the scatterer itself. There-

fore, we will represent the dielectric cylinders located at the
points \((x_a,z_a)\) by single 2D electric dipoles:

\[ \mathcal{P}(x,z) = \sum_{a=1}^{N} p_a \delta^2(x-x_a,z-z_a). \]  

In the present model we place the dielectric cylinders \((3)\)
to between two infinite, perfectly conducting mirrors described
by the equations \(x = 0\) and \(x = d\). For simplicity, we consider
only the case where the cylinders are oriented parallel to the
mirrors. Moreover, our discussion will be restricted to the
frequencies from the following range:

\[ \pi < k d < 2 \pi, \]  

where \(k = \omega/c\) is the wave number in vacuum. Thus in the

plane waveguide formed by the two parallel mirrors sepa-
rated by a distance \(d\), only one guided TE mode exists [22]:

\[ \mathcal{E}^{(0)}(x,z) = \frac{2}{\sqrt{\beta d}} \sin(\alpha x) e^{i\beta z}, \]  

where the propagation constants are given by

\[ \alpha = \frac{\pi}{d}, \quad \beta = \sqrt{k^2 - \alpha^2}. \]  

A brief discussion of other cases is given in the final section
of this paper.

The total field that can be measured \textit{far} from the cylinders
is fully described by the reflection \(\rho\) and transmission \(\tau\) co-
efficients:

\[ \mathcal{E}(x,z) = \begin{cases} \mathcal{E}^{(0)}(x,z) + \rho \mathcal{E}^{(0)*}(x,z) & \text{for } z \to -\infty \\ \tau \mathcal{E}^{(0)}(x,z) & \text{for } z \to +\infty. \end{cases} \]  

Using the Lorentz theorem and repeating the straightforward
but lengthy calculations (see, e.g., [22]) we easily arrive at
the following expressions determining the transmission coef-
ficient:

\[ \tau = 1 + i \pi \frac{k^2}{\alpha} \sum_{a=1}^{N} p_a \mathcal{E}^{(0)*}(x_a,z_a) \]  

and the reflection coefficient

\[ \rho = i \pi \frac{k^2}{\alpha} \sum_{a=1}^{N} p_a \mathcal{E}^{(0)}(x_a,z_a), \]  

for given dipole moments \(p_a\). In the following section we
will relate \(p_a\) to the values of the incident field calculated
at the positions of the cylinders \(\mathcal{E}^{(0)}(x_a,z_a)\).

III. METHOD OF IMAGES

A simple way to take into account the boundary condi-
tions of parallel mirrors and their influence on the electro-
magnetic field is to use the method of images. This technique
has been used, i.e., in QED calculations of spontaneous
emission in cavities [23,24]. To reproduce the correct bound-
ary conditions on the radiation field of each cylinder \((3)\), the
mirrors are replaced by an array of image cylinders whose phases alternate in sign:

\[ \mathcal{P}(x,z) = \sum_{a=1}^{N} \sum_{j=-\infty}^{\infty} (-1)^j p_a \delta^2(x-x_a^j, z-z_a) \]

where

\[ x_a^j = (-1)^j x_a + jd. \]

Thus a finite system of dielectric cylinders (3) placed within a metallic waveguide is fully equivalent to an infinite system of cylinders (10) forming a slab in a free space. This fact allows us to utilize some results from our previous paper concerning dielectric cylinders in free space [17].

It is now a well-established fact that to use safely the point-scatterer approximation it is essential to use a representation for the cylinders that conserves energy in the scattering processes. In the case of a system of cylinders in free space, this requirement gives the following form of the coupling between the dipole moment \( p_a \) and the electric field incident on the cylinder \( E'(x_a, z_a) \) [17]:

\[ i\pi k^2 p_a = \frac{e^{i\phi} - 1}{2} E'(x_a, z_a). \]

The same result holds also for a system of cylinders placed in a metallic waveguide (10). In this case the field acting on the \( a \)th cylinder,

\[ E'(x_a, z_a) = E^{(0)}(x_a, z_a) + \frac{e^{i\phi} - 1}{2} \sum_{b=1}^{N} G_{ab} E'(x_b, z_b), \]

is the sum of the incident guided mode \( E^{(0)} \), which obeys the Maxwell equations in an empty waveguide, and waves scattered by all other cylinders and all images. Thus, in the present model the \( G \) matrix from Eq. (13) needs to be defined differently than in Ref. [17]:

\[ i\pi G_{ab} = 2 \sum_{p_{ab}^{(j)} \neq 0} (-1)^j K_0(-ikp_{ab}^{(j)}), \]

where

\[ p_{ab}^{(j)} = \sqrt{(x_a-x_{a}^{(j)})^2+(z_a-z_{b})^2} \]

denotes the distance between the \( a \)th cylinder and the \( j \)th image of the \( b \)th cylinder and \( K_0 \) is the modified Bessel function of the second kind. Note that summation in Eq. (14) is performed over all \( j \), for which \( p_{ab}^{(j)} \neq 0 \).

The system of linear equations (13) fully determines the field acting on each cylinder \( E'(x_a, z_a) \) for a given field of the guided mode \( E^{(0)}(x_a, z_a) \) incident on the system. Analogous relationships between the stationary outgoing wave and the stationary incoming wave are known in the general scattering theory as the Lippmann-Schwinger equations [25]. If we solve Eqs. (13) and use Eqs. (12) to find \( p_a \), then we are able to find the transmission and reflection coefficients given by Eqs. (8) and (9).

**FIG. 1.** Transmission \( T \) of the system of dielectric cylinders described by the phase shifts \( \phi = -1 \) placed randomly in a planar metallic waveguide plotted as a function of the number of cylinders \( N \).

**IV. TRANSMISSION EXPERIMENT**

The actual properties of physical systems have to be observed experimentally and it is not enough just to know the properties of the stationary solutions of the Maxwell equations. These are only theoretical tools. Experiments deal rather with measurable quantities and, for many practical problems, a natural quantity to look for is the transmission \( T = |r|^2 \) of a finite system of characteristic size \( L \) and its dependence on \( L \). As a simple example let us consider a system of \( N \) cylinders placed between the mirrors separated by a distance \( k d = 3\pi/2 \). The cylinders are distributed randomly with constant uniform density \( n = 1 \) cylinder per wavelength squared. Therefore for each \( N \) the size of the system is proportional to the number of cylinders \( N \times N \). The images of the cylinders from Eq. (10) were summed from \( j = -50000 \) to \( j = 50000 \). In Fig. 1 we present the log-linear plot of the transmission \( T \) as a function of the number of cylinders \( N \) calculated for a fixed value of \( \phi = -1 \). It follows from inspection of this figure that \( T \propto e^{-L/k} \). This proves that in the limit \( L \to \infty \) our system indeed supports a localized state and behaves as an optical insulator. Consequently, Anderson localization occurs when this happens.

In Fig. 2 we plot the transmission \( T \) of the systems of \( N = 1, 10, 100 \) cylinders as a function of the phase shift of a single cylinder \( \phi \). We see that in the case \( N = 1 \) the incident wave is totally reflected for a single value of \( \phi = \phi_0 \). Note that not necessarily \( |\phi_0| = \pi \), and therefore for this value of \( \phi \) the total scattering cross section \( \sigma \) of an individual dielectric cylinder [17,18],

\[ k\sigma = 2(1 - \cos \phi), \]

does not approach its maximal value. However, for systems containing \( N = 10 \) and \( N = 100 \) the entire regions of the values of phase shifts \( \phi \) exist for which \( T = 0 \). They are separated by narrow maxima of transmission. Moreover, inspection of Fig. 2 suggests that in the limit \( N \to \infty \) the number of these maxima increases and simultaneously they became narrower and sharper. Therefore, we may expect that for suffi-
corresponds to the case when the polarization of the medium is equal to unity. It can be shown that this dense set of the systems of dielectric cylinders placed in a planar metallic waveguide correspond to nonzero solutions $\mathcal{E}_l(x_a,z_a)$ of Eqs. (13) in the right-hand side of Eq. (5). As the considered medium is nondissipative, the time average energy stream integrated over a closed surface surrounding the medium must vanish. This means that there are again no guided waves in the radiation field. From the preceding section, the condition $|\rho|^2 = 0$ holds. Using Eq. (12), we see that the vector formed by the values of the field acting on the cylinders is orthogonal to the vector formed by the values of incident field calculated at the positions of the cylinders:

$$\sum_{a=1}^{N} \mathcal{E}_l(x_a,z_a) \mathcal{E}^{(0)\dagger}(x_a,z_a) = 0.$$  

But simultaneously $\mathcal{E}_l(x_a,z_a)$ is a solution of a system of linear Eqs. (13) where $\mathcal{E}^{(0)}(x_a,z_a)$ is the right-hand side. Therefore $\mathcal{E}_l(x_a,z_a)$ is also a solution of Eqs. (13) with $\mathcal{E}^{(0)}(x_a,z_a) = 0$. Note that in this case the latter system of equations is equivalent to the eigenproblem for the $G_{ab}$ matrix:

$$\sum_{b=1}^{N} G_{ab} \mathcal{E}_l(x_b,z_b) = \lambda_l \mathcal{E}_l(x_a,z_a),$$

where

$$\frac{1}{\lambda_l} = e^{i\phi_l} - \frac{1}{2}.$$  

The proof works also the other way round. Suppose that $\mathcal{E}_l(x_a,z_a)$ is a solution of Eqs. (13) for $\mathcal{E}^{(0)}(x_a,z_a) = 0$. As the considered medium is nondissipative, the time average energy stream integrated over a closed surface surrounding the medium must vanish. This means that there are again no guided modes in the radiation field [which in the case $\mathcal{E}^{(0)}(x,z) = 0$ is equal to the total field]. Therefore Eq. (17) holds and the wave is localized.

Let us stress that Eq. (20) can be fulfilled only if the real part of an eigenvalue satisfies

$$\text{Re} \lambda_l = -1.$$  

The imaginary part of the eigenvalue and the phase shift are then related by

$$\tan \frac{\phi_l}{2} = -\frac{1}{\text{Im} \lambda_l}.$$  

Therefore only those eigenvectors $\mathcal{E}_l(x_a,z_a)$ of the $G_{ab}$ matrix which correspond to the eigenvalues $\lambda_l$ satisfying the condition (21) may be related to localized waves. Moreover, those waves can exist only if the phase shift $\phi$ which determines the scattering properties of the cylinders is equal to the
Thus the values of $kd$ thickness of the waveguide points of the functions supported by Maxwell’s equations in two dimensions frequencies and the dense band of localized waves. Therefore, a distinction may be provided by investigation of a phase transition which occurs in the limit of $N \to \infty$ in the spectra of $G_{ab}$ matrices corresponding to systems of randomly distributed dielectric cylinders.

To support this statement in Fig. 3 we plot the spectrum $\lambda_1$ of a $G$ matrix (diagonalized numerically) corresponding to a certain specific configuration of $N=100$ cylinders placed randomly with the uniform density $n=1$ cylinder per wavelength squared. We see that quite a lot of eigenvalues are located near the Re $\lambda = -1$ axis. As will be discussed below, this is a universal property of 2D $G$ matrices, not restricted to this specific realization of the system only. To prove these statements we diagonalize numerically the $G$ matrix (14) for $10^2$ different distributions of $N=100$ cylinders.

This means that in this limit for almost any random distribution of the cylinders, an infinite number of eigenvalues satisfies the condition Eq. (21). It is therefore reasonable to expect that in the case of a random and infinite system a countable set of frequencies $\omega_l$ corresponding to localized waves becomes dense in some finite interval. But it is always difficult to separate such frequencies from frequencies which may be arbitrarily near and physically the spectrum is always a coarse-grained object. Therefore in the limit of an infinite medium an entire band of spatially localized electromagnetic waves appears.

VI. CONCLUDING REMARKS

Anderson localization of electromagnetic waves in random arrays of $N$ dielectric cylinders confined within a planar metallic waveguide of thickness $d$ has been studied. In studying the properties of the stationary solutions of the Maxwell equations in such two-dimensional media, several particular cases may be considered ($kd=\pi$ is the cutoff thickness of the waveguide). For $N=0$ and $kd<\pi$ there are no guided
modes in the waveguide as well as no localized waves. This case is analogous to the electronic band gap in a solid. If \( N > 0 \) and \( k d < \pi \) there are still no guided modes in the waveguide but localized waves can appear for any distribution of the cylinders. It is again analogous to the solid state physics situation where isolated perturbations of the periodicity of crystals (like impurities or lattice defects) can lead to the formation of localized electronic states with energies within the forbidden band. Another possibility corresponds to \( N = 0 \) and \( k d > \pi \). In this case there are guided modes but the system supports no localized waves. This is very similar to the conductance band in solids. Guided modes correspond to extended electronic states described by Bloch functions. In this paper we have performed a detailed study of the regime where \( N > 0 \) and \( k d > \pi \). For this range of parameters there are both guided modes and resonances of transmission. Isolated localized waves can be seen for certain distributions of the cylinders. The signs of Anderson localization emerging in the limit of an infinite medium can be observed both in analysis of transmission and in the properties of the spectra of certain random matrices. Eventually let us consider a limiting case of \( N \to \infty \) and \( k d > \pi \). Now the guided modes no longer exist in the waveguide. Instead a band of localized waves will be formed for any distribution of the cylinders. It is an interesting analog of the Anderson localization in non-crystalline solids such as amorphous semiconductors or disordered metals.

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