Exact results for density grain statistics in the interacting 1d Bose gas

Piotr Deuar

Joanna Pietraszewicz

Institute of Physics, Polish Academy of Sciences, Warsaw, Poland

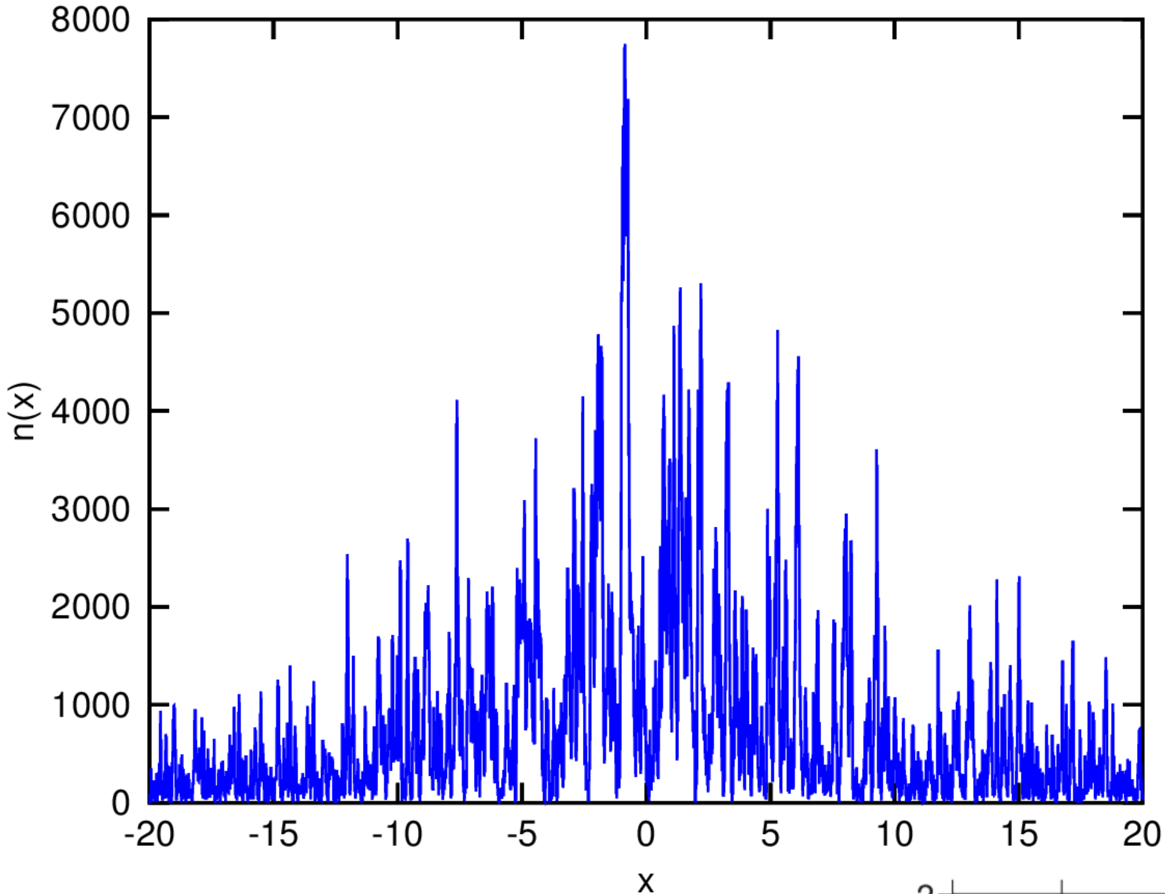
New J. Phys (2017), arXiv:1708.00031

Outline

- Density fluctuations and what is actually measured *in situ*
- New equations derived from Yang-Yang exact solution
- Statistics of number fluctuations (Poissonian / not)
- Size of independently fluctuating density grains
- Distribution of density grain size (skewness, kurtosis)
- Experimental suggestion

Local density fluctuations

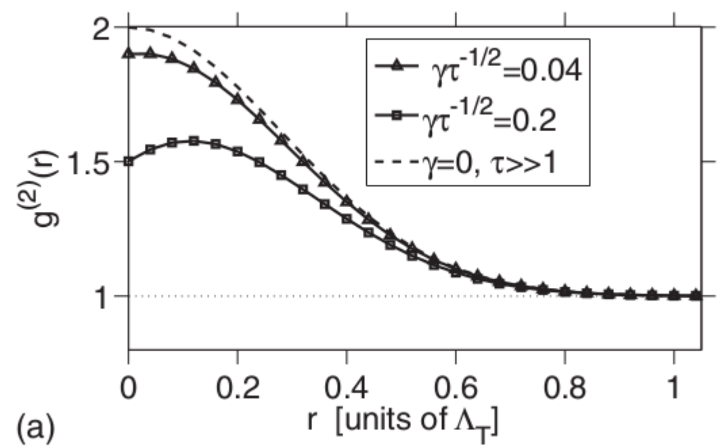
- Single shot



classical Field at relatively high T

- Ensemble nicely characterised by normalised density-density correlation

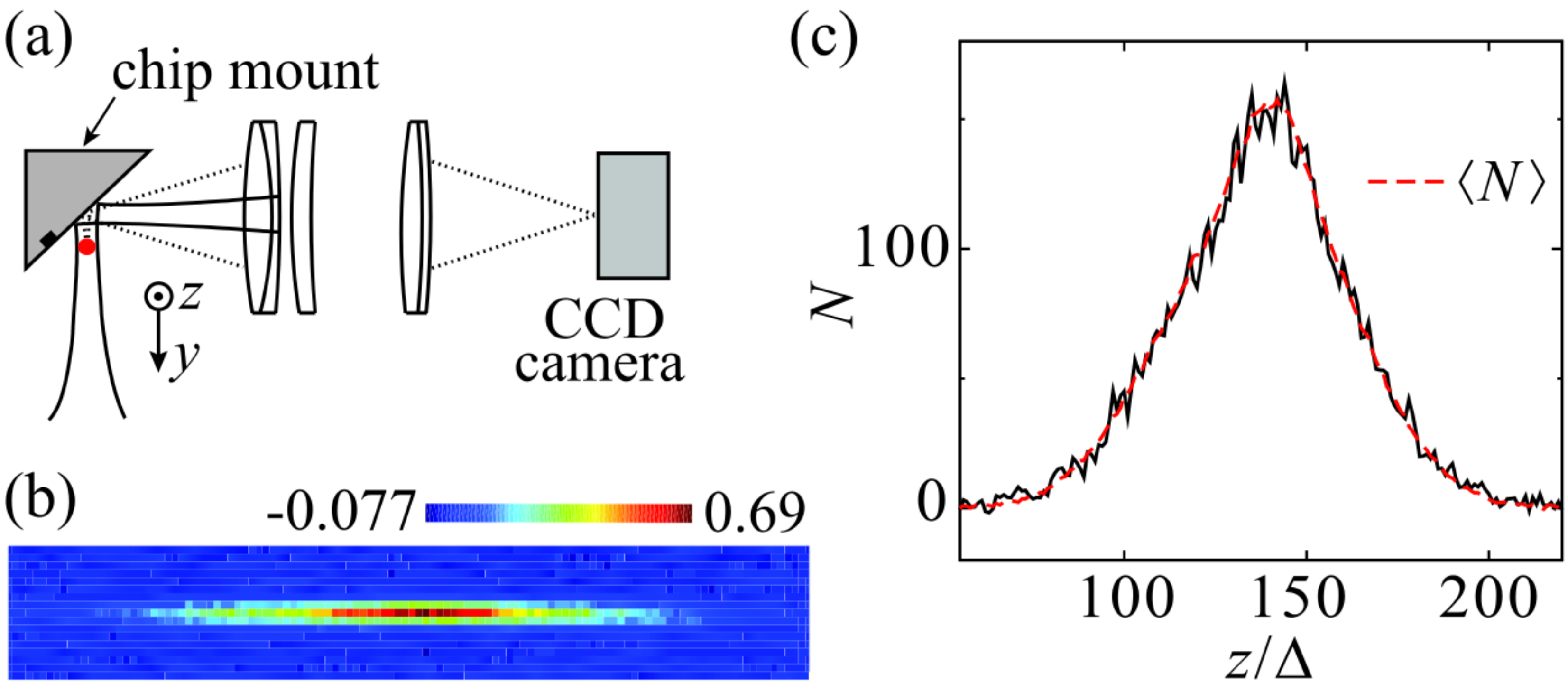
$$g^{(2)}(z) = \frac{1}{n^2} \langle \widehat{\Psi}^\dagger(x) \widehat{\Psi}^\dagger(x+z) \widehat{\Psi}(x+z) \widehat{\Psi}(x) \rangle$$



(a)

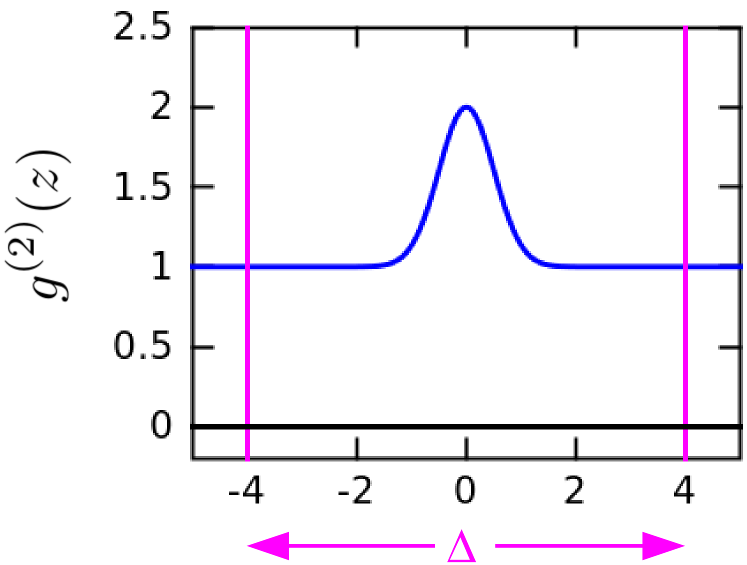
*PD, Sykes, Gangardt, Davis, Drummond, Kheruntsyan, PRA **79**, 043619 (2009)*

Experimental view

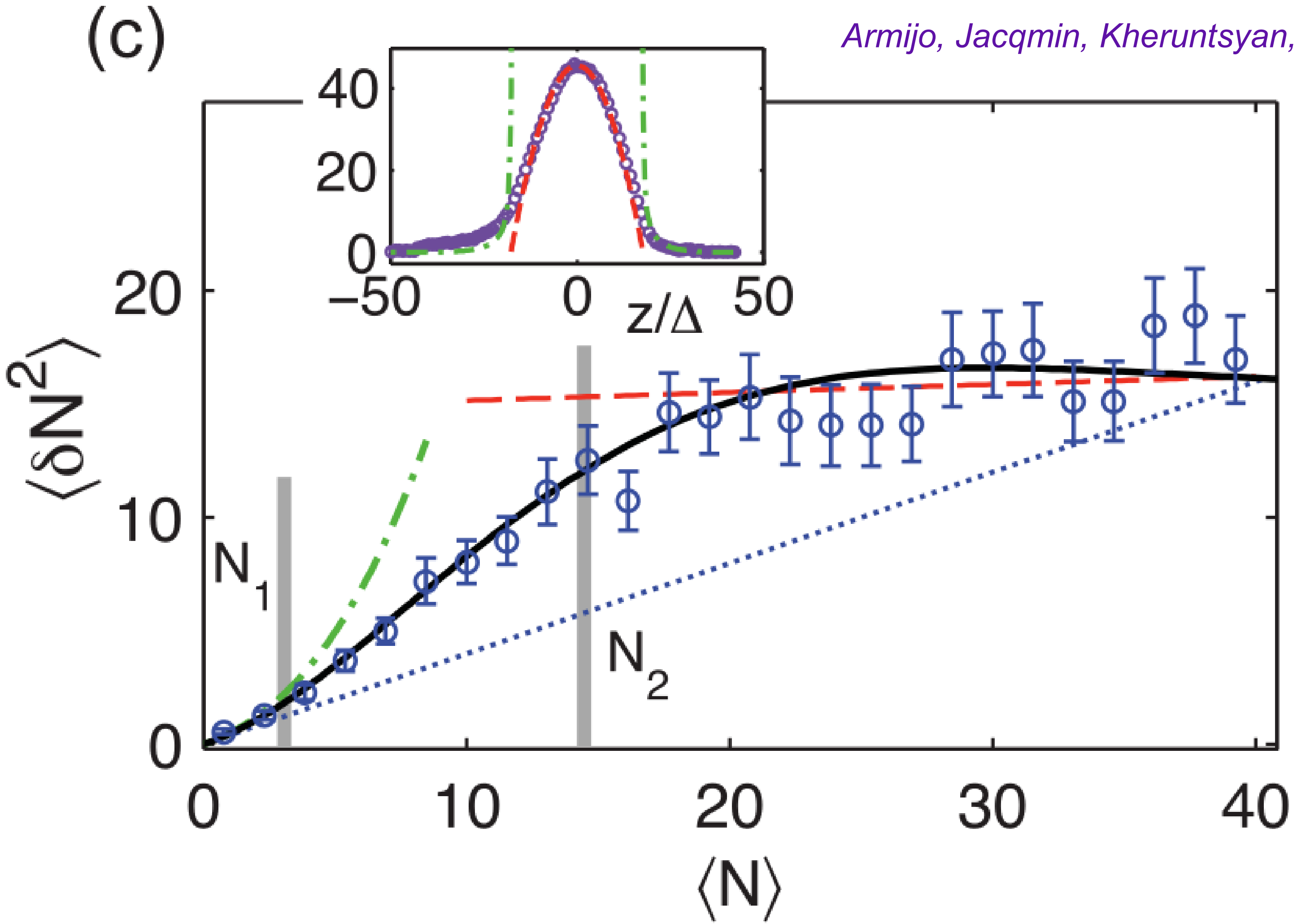


Armijo, Jacqmin, Kheruntsyan, Bouchoule, PRL ***105****, 230402 (2010)*

Bin count



Resolution is insufficient to resolve *$g^{(2)}(z)$*



Armijo, Jacqmin, Kheruntsyan, Bouchoule, PRA ***83****, 021605 (2011)*

Observable fluctuations are well characterised by

$$S_0 = \frac{\mathrm{var}\, N}{N}$$

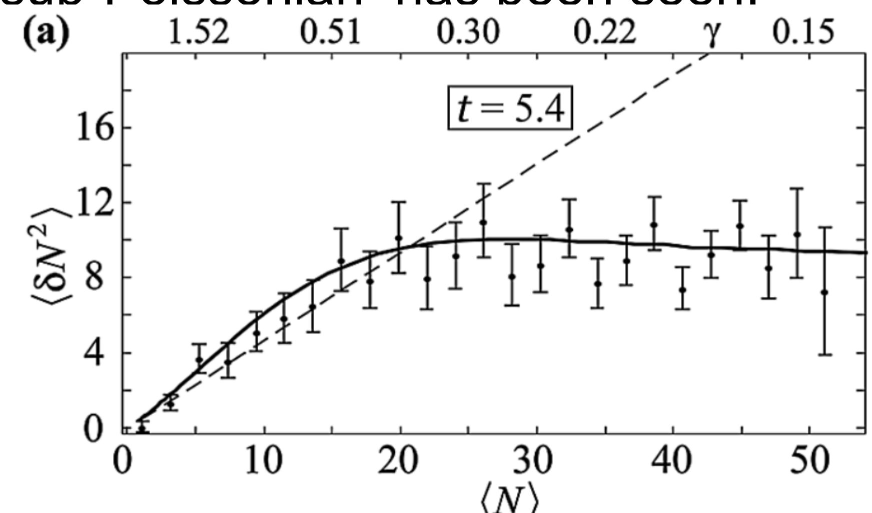
Coarse-grained fluctuations

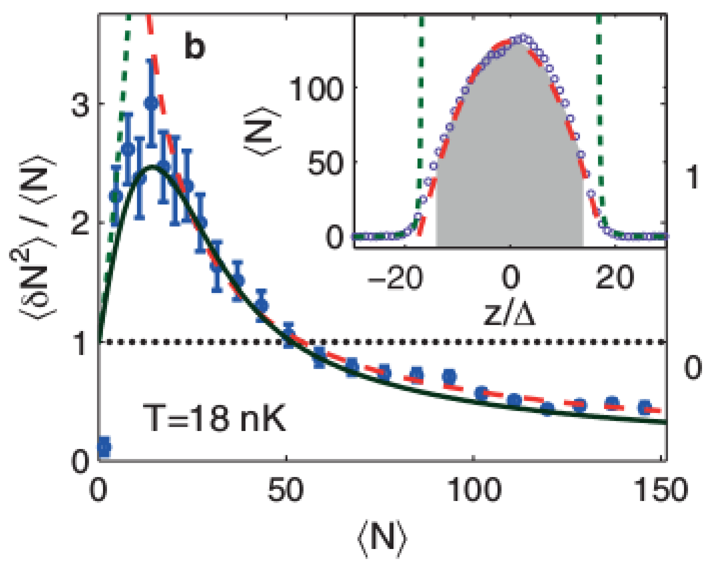
$$S_0 = \frac{\mathrm{var}\, N}{N}$$

- Related to the local fluctuations by

$$S_0 = 1 + n \int_{\text{system}} dz \left[g^{(2)}(z) - 1 \right]$$

- S_0 = 1 Poissonian statistics (BEC, high T, shot noise)
- S_0 > 1 super-Poissonian bunches
- S_0 < 1 sub-Poissonian
- sub-Poissonian has been seen:





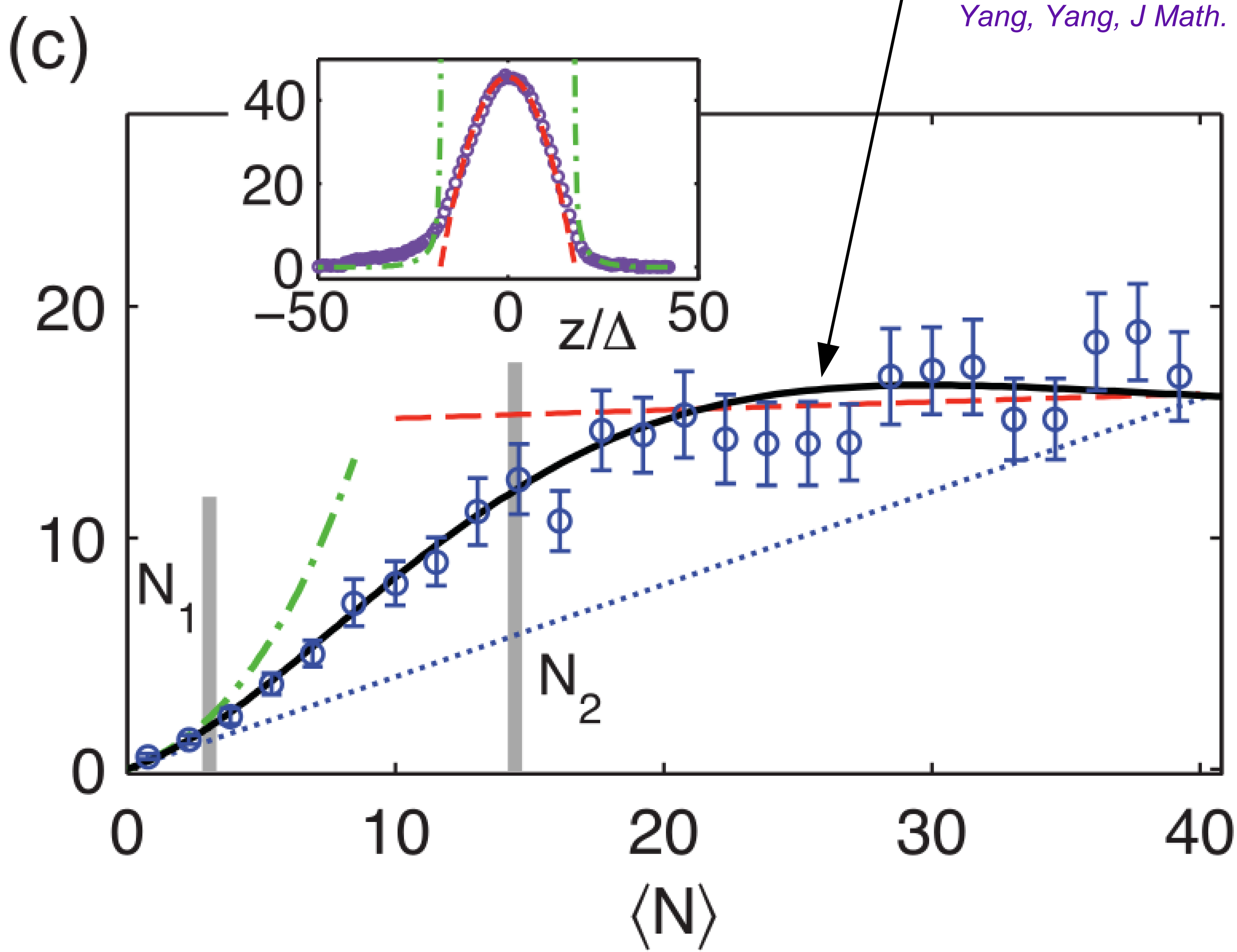
*Jacqmin, Armijo, Berrada, Kheruntsyan, Bouchoule, PRL **106**, 230405 (2011)*

*Armijo, PRL **108**, 225306 (2012)*

Theory for S_0

- Yang-Yang exact solution in a local density approximation for each bin

Yang, Yang, J Math. Phys. ***10****, 1115 (1969)*



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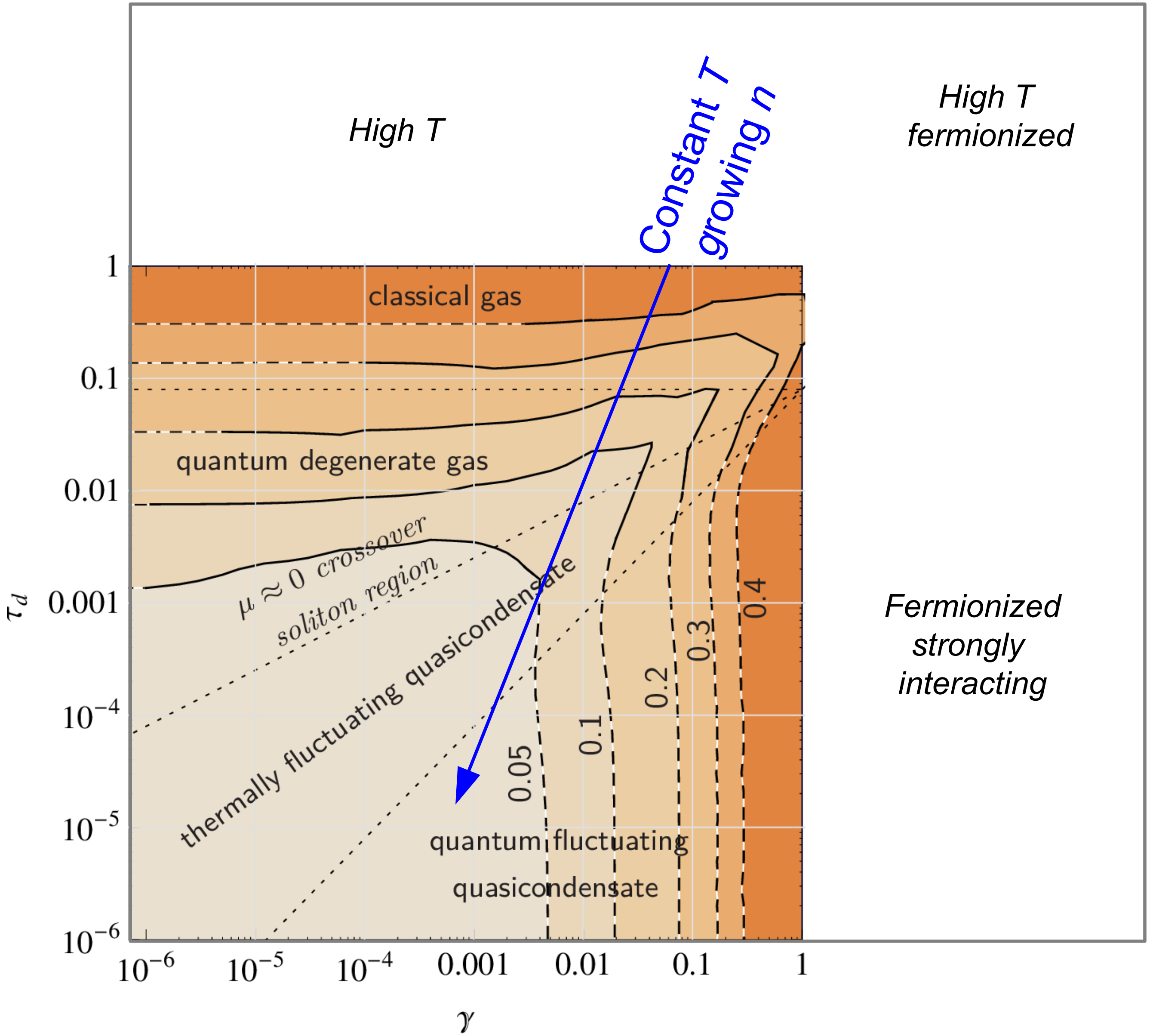
Physical parameters



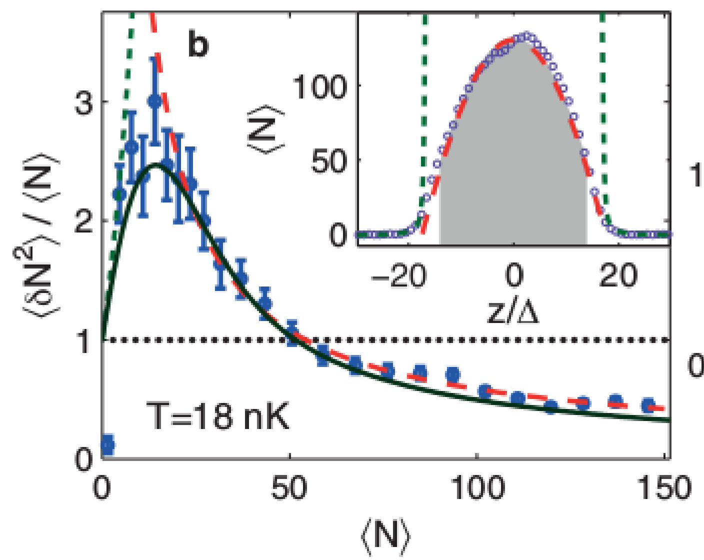
adapted from Pietraszewicz, PD, arXiv:1707.01776

Interaction strength

$$\gamma = \frac{mg}{\hbar^2 n}$$

Relative temperature

$$\tau = \frac{2mk_BT}{\hbar^2 n^2} = \frac{4\pi T}{T_d}$$



Yang-Yang theory gives density

- By solving a bunch of Fredholm integral equations
- Spectrum of quasimomenta

$$\varepsilon(k) = -\mu + \frac{k^2}{2} - \frac{gT}{\pi}\int_{-\infty}^{\infty} \frac{dq\ \log\left[1+e^{-\varepsilon(q)/T}\right]}{g^2+(k-q)^2}$$

Iteration start

$$\varepsilon^{(0)}(k) = -\mu + \frac{k^2}{2}$$

- Density of occupied quasimomenta

$$2\pi\rho(k)\left[1+e^{\varepsilon(k)/T}\right] = 1 + 2g\int_{-\infty}^{\infty}\frac{dq\ \rho(q)}{g^2+(k-q)^2}$$

Iteration start

$$\rho^{(0)}(k) = \frac{1}{2\pi\left[1+e^{\varepsilon(k)/T}\right]}$$

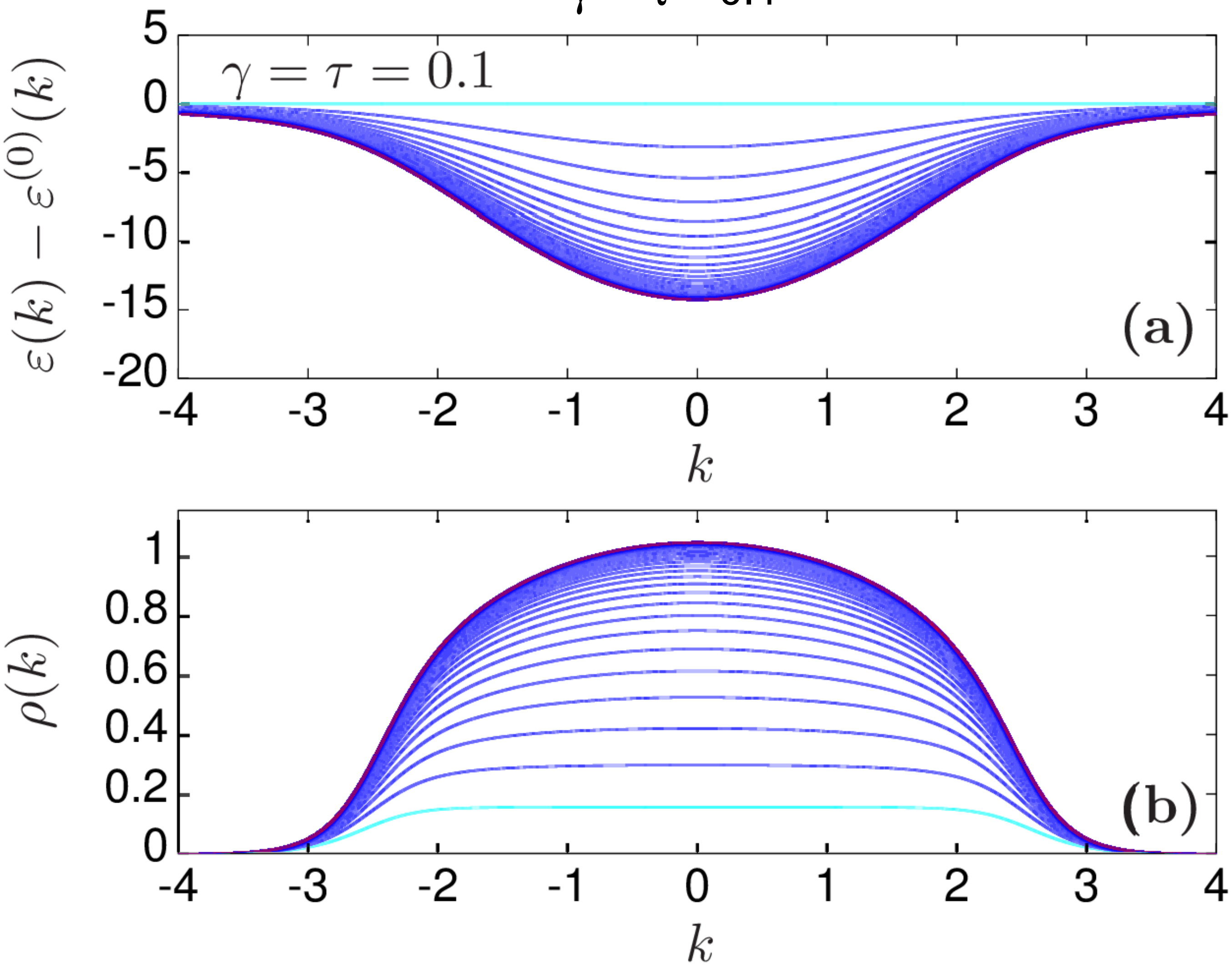
- Particle density

$$n = \frac{N}{L} = \frac{\langle\widehat{N}\rangle}{L} = \int_{-\infty}^{\infty}\rho(k)dk$$

Iteration examples

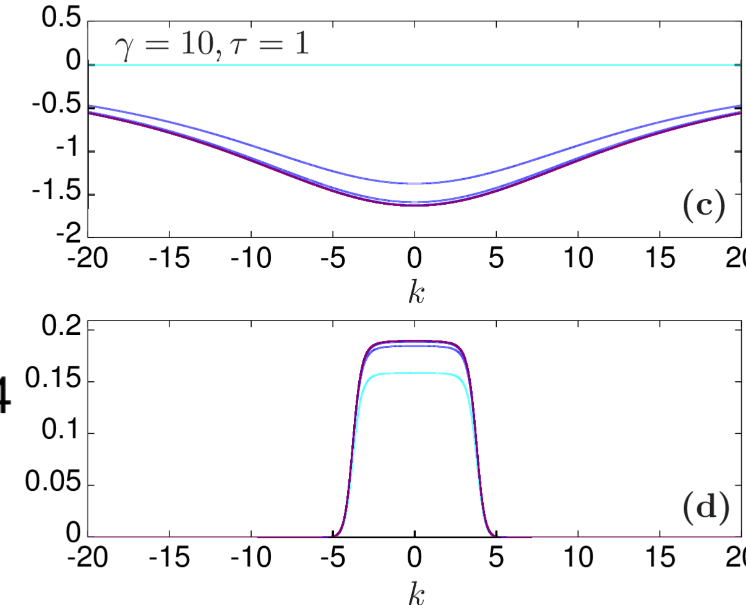
Quasicondensate

γ = τ = 0.1



Fermionized

γ = 10 τ = 1



$$\varepsilon^{(0)}(k) = -\mu + \frac{k^2}{2}$$

Density gradient is related to S_0

- Partition function

$$\widehat{Z} = \exp\left[-\left(\widehat{H}-\mu\widehat{N}\right)/T\right] \qquad \mathcal{Z} = \mathrm{Tr}[\widehat{Z}]$$

- Moments of total number *N*

$$\langle\widehat{N}\rangle = \frac{\mathrm{Tr}[\widehat{N}\widehat{Z}]}{\mathcal{Z}} = \frac{k_B T}{\mathcal{Z}}\left(\frac{\partial\mathcal{Z}}{\partial\mu}\right)_{g,T} \qquad \langle\widehat{N}^2\rangle = \frac{(k_B T)^2}{\mathcal{Z}}\left(\frac{\partial^2\mathcal{Z}}{\partial\mu^2}\right)_{g,T}$$

- Fluctuations

$$\delta\widehat{N} = \widehat{N} - \langle\widehat{N}\rangle \qquad \langle\delta\widehat{N}^2\rangle = \mathrm{var}N = k_B T\left(\frac{\partial\langle\widehat{N}\rangle}{\partial\mu}\right)_{g,T}$$

- Result

$$S_0 = \frac{\mathrm{var}N}{N} = \frac{k_B T}{n}\left(\frac{\partial n}{\partial\mu}\right)_{g,T}$$

Approximation by finite-difference

- The usual method:

$$S_0 \approx \frac{k_B T}{n(\mu)} \left[\frac{n(\mu + \frac{1}{2}\Delta\mu) - n(\mu - \frac{1}{2}\Delta\mu)}{\Delta\mu} \right]$$

- Choose small $\Delta\mu$
- Calculate n at several values, by solving the integral equations

- BUT: doesn't work very accurately for many parameter regimes

Alternative method:

$$S_0 = \frac{k_B T}{n}\left(\frac{\partial n}{\partial \mu}\right)_{g,T} \qquad \text{and} \qquad n = \int_{-\infty}^{\infty} \rho(k) dk$$

- So, calculate S_0 from a new kernel

$$S_0 = \frac{\mathrm{var} N}{N} = \frac{T}{n}\int_{-\infty}^{\infty} \rho'(k) dk \qquad \rho'(k) = \frac{\partial \rho(k)}{\partial \mu}$$

- It obeys a new equation:

$$\rho'(k)\left[1 + e^{\varepsilon(k)/T}\right] + \frac{\rho(k)\varepsilon'(k)}{T} e^{\varepsilon(k)/T} = \frac{g}{\pi}\int_{-\infty}^{\infty} \frac{dq\ \rho'(q)}{g^2 + (k-q)^2}$$

- Which depends on another quantity $\varepsilon'(k) = \frac{\partial \varepsilon(k)}{\partial \mu}$

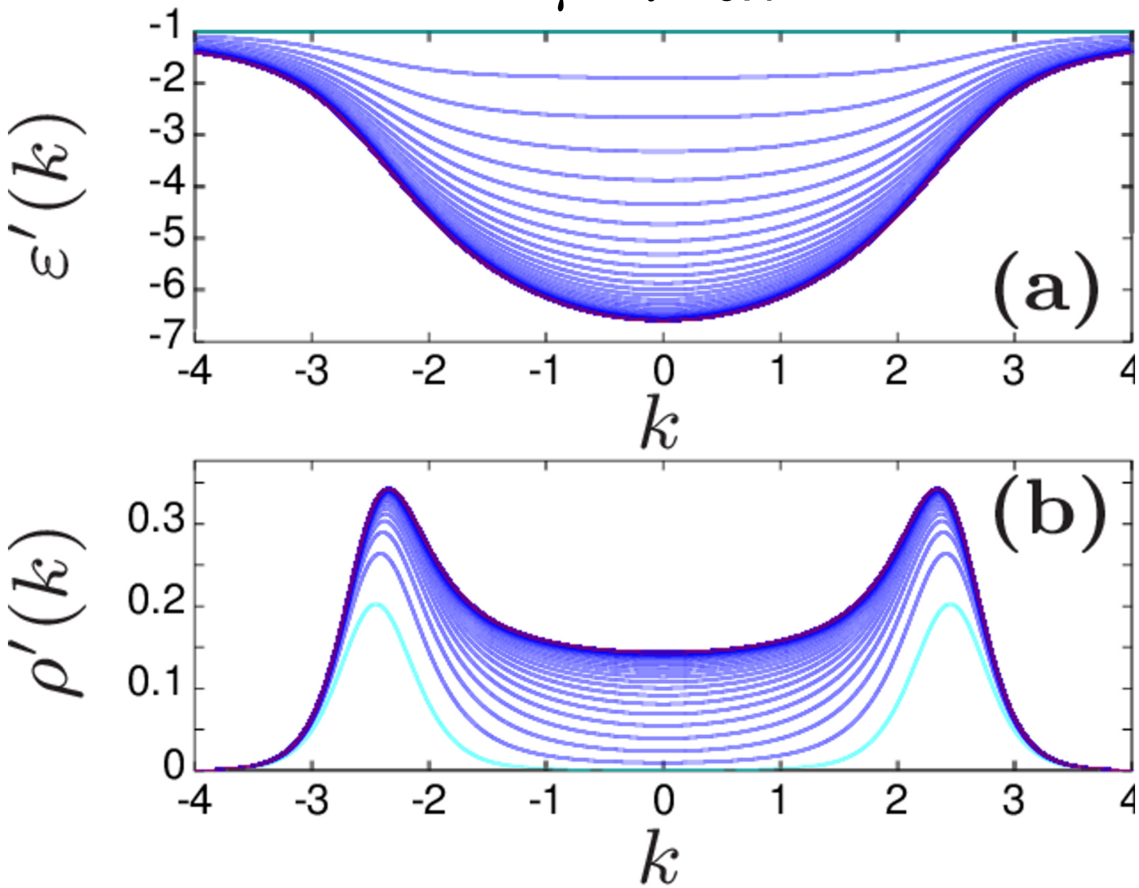
$$\varepsilon'(k) = -1 + \frac{g}{\pi}\int_{-\infty}^{\infty} \frac{dq\, \varepsilon'(q)}{g^2 + (k-q)^2} \frac{1}{1 + e^{\varepsilon(q)/T}}$$

- Start iterations with:

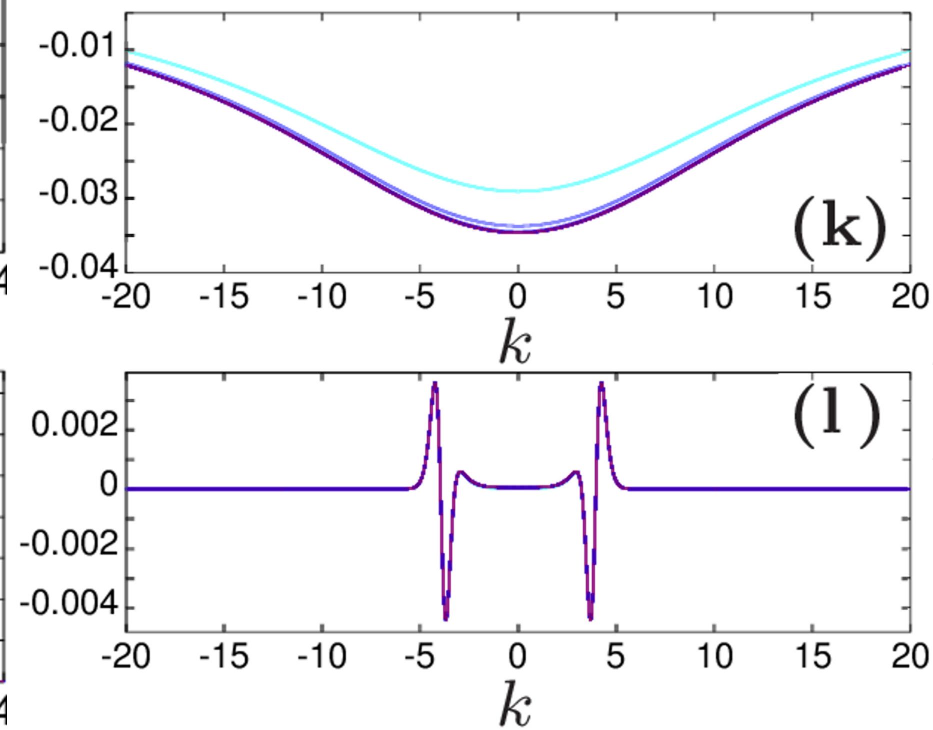
$$\varepsilon'^{(0)}(k) = -1 \qquad \rho'^{(0)}(k) = -\frac{1}{T}\frac{\rho(k)\varepsilon'(k)}{1 + e^{-\varepsilon(k)/T}}$$

Iterations work nicely

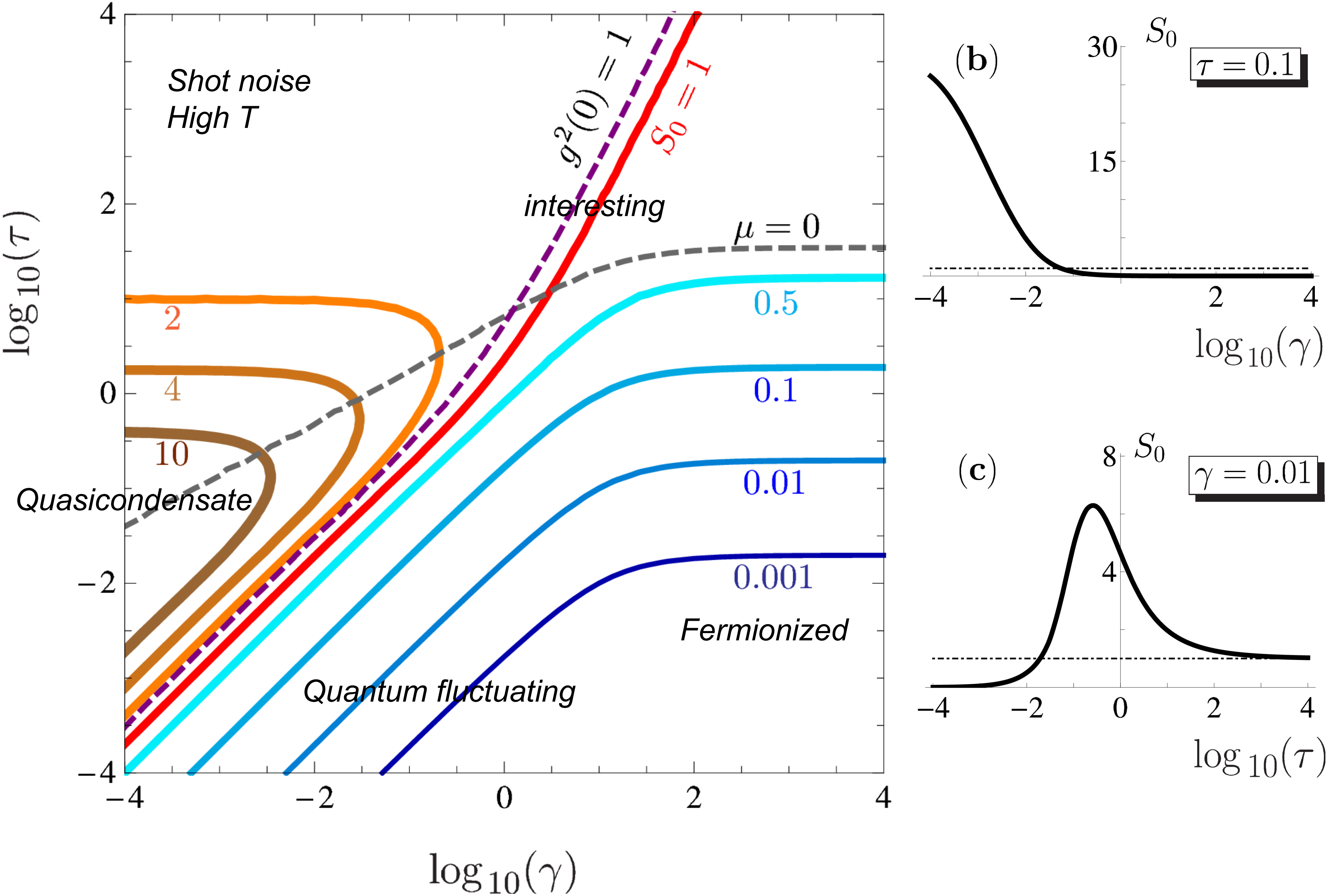
$\gamma = \ \tau = 0.1$



$\gamma = 10 \ \tau = 1$



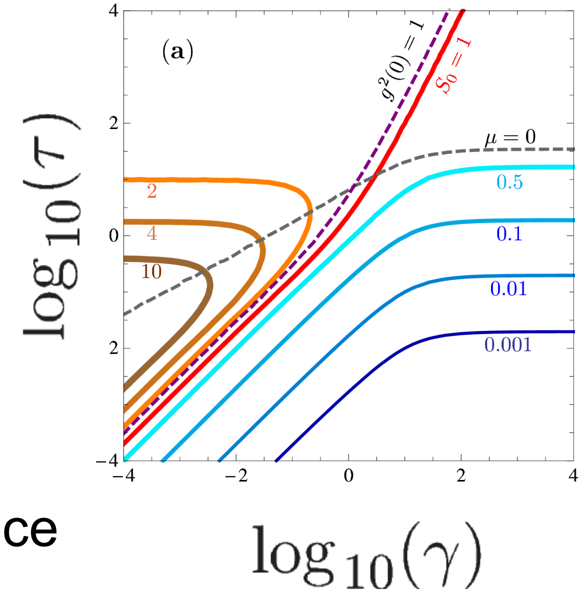
Phase diagram for bin statistics S_0



Comments

- Classical field / quasicondensate region has a bulge with $S_0 >> 1$

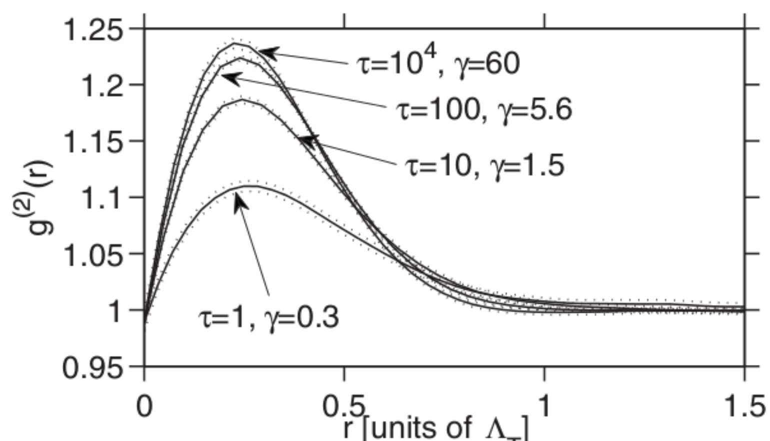
$$S_0 \approx \frac{k_B T}{\mu} \quad \text{here}$$



- Fermionized regime has $S_0 << 1$
- Low temperature with $k_B T << \mu$ always has $S_0 << 1$
- Classical thermal gas has shot noise $S_0 = 1$
- $g^{(2)}(0) \leftrightarrow$ $S_0 > 1$ indicates a hump in $g^{(2)}(z)$ at nonzero z since

$$S_0 = 1 + n \int_{\text{system}} dz \left[g^{(2)}(z) - 1 \right]$$

→ confirms earlier studies



*PD, Sykes, Gangardt, Davis, Drummond, Kheruntsyan, PRA **79**, 043619 (2009)*

Density grains

- Now lets think about localized independent fluctuations.
- Suppose the total number of particles in the gas comes from *p* independent contributions (DENSITY GRAINS)

$$N = \langle \widehat{N} \rangle = \sum_{j=1}^{p} \langle \widehat{\mathcal{N}}_j \rangle = p\overline{\mathcal{N}}$$

- The occupation of the *j* th grain in some realization is $\mathcal{N}_j$
- The mean density grain occupation is $\overline{\mathcal{N}} = \frac{1}{p}\sum_{j=1}^{p} \langle \widehat{\mathcal{N}}_j \rangle$
- The second moment: $\langle \widehat{N}^2 \rangle = \sum_{jj'} \langle \widehat{\mathcal{N}}_j \widehat{\mathcal{N}}_{j'} \rangle = \sum_{j} \langle \widehat{\mathcal{N}}_j^2 \rangle + \sum_{j \neq j'} \langle \widehat{\mathcal{N}}_j \rangle \langle \widehat{\mathcal{N}}_{j'} \rangle$

$$= p\left[\overline{\mathcal{N}^2} + (p-1)\overline{\mathcal{N}}^2 \right].$$

$$\mathrm{var}N = p\left[\overline{\mathcal{N}^2} - \overline{\mathcal{N}}^2 \right] = p\,\mathrm{var}\mathcal{N}$$

- The statistics $S_0 = \frac{\mathrm{var}N}{N} = \frac{\mathrm{var}\mathcal{N}}{\mathcal{N}}$ is the SAME for grains and total N

Intensive quantities

- There are other intensive quantities related to the distribution of N and $\mathcal{N}$
- They take the same form in N and $\mathcal{N}$

$$S_0 = \frac{\langle \delta \widehat{N}^2 \rangle}{\langle \widehat{N} \rangle} = \frac{\overline{\delta \mathcal{N}^2}}{\overline{\mathcal{N}}},$$

$$M_3 = \frac{\langle \delta \widehat{N}^3 \rangle}{\langle \widehat{N} \rangle} = \frac{\overline{\delta \mathcal{N}^3}}{\overline{\mathcal{N}}},$$

$$M_4 = \frac{\langle \delta \widehat{N}^4 \rangle - 3\langle \delta \widehat{N}^2 \rangle^2}{\langle \widehat{N} \rangle} = \frac{\overline{\delta \mathcal{N}^4} - 3\left(\overline{\delta \mathcal{N}^2}\right)^2}{\overline{\mathcal{N}}}$$

Properties of the distribution

- Skewness $s = \frac{\langle \delta \widehat{N}^3 \rangle}{(\mathrm{var} N)^{3/2}} = \frac{M_3}{S_0^{3/2}\sqrt{N}}$ (not intensive)

- Skewness of density grain distribution

$$s_{\mathcal{N}} = \frac{\overline{\delta \mathcal{N}^3}}{(\overline{\delta \mathcal{N}^2})^{3/2}} = s\sqrt{p} = \frac{M_3}{S_0^{3/2}\sqrt{\overline{\mathcal{N}}}}$$

(intensive)

- While intensive, we don't yet know the mean density grain occupation $\overline{\mathcal{N}}$

- Kurtosis $\kappa = \frac{\langle \delta \widehat{N}^4 \rangle}{(\mathrm{var} N)^2}$

$$\kappa_{\mathcal{N}} = \frac{\overline{\delta \mathcal{N}^4}}{(\overline{\delta \mathcal{N}^2})^2} = 3 + \frac{M_4}{S_0^2\,\overline{\mathcal{N}}}.$$

$$\kappa = 3 + \frac{\overline{\mathcal{N}}}{N}\left[\kappa_{\mathcal{N}} - 3\right]$$

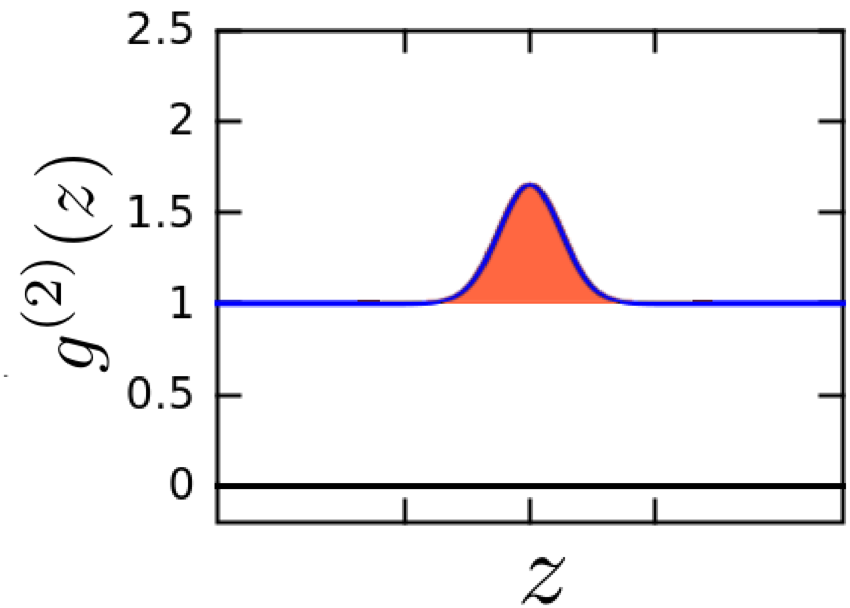
Size of density grain

- Particles inside an independent density grain should be correlated
- Consider the peak of two-body density correlations
  - Width *w* of the $g^{(2)}(z)$ gives range over which particles are correlated
  - Height $h = g^{(2)}(0)-1$ quantifies the amount/probability of fluctuation away from the mean
- Taking *width* x *density* does not give good results $\overline{\mathcal{N}} = wn?$
  - Condensate, thermal state both get stupid values
- Need to also include the "first" particle in the correlation

- A well behaving estimate:

$$n\int\left[g^{(2)}(z)-1\right]\,dz+1$$

- So: $\overline{\mathcal{N}} \approx S_0$



Predictions for density grain properties

- Taking the value

$$\overline{\mathcal{N}} = S_0$$

- Also lets us evaluate skewness and kurtosis of density grain distributions

$$s_{\mathcal{N}} = \frac{\overline{\delta\mathcal{N}^3}}{(\overline{\delta\mathcal{N}^2})^{3/2}} = \frac{M_3}{S_0^{3/2}\sqrt{\overline{\mathcal{N}}}} = \frac{M_3}{S_0^2}$$

> 0 tail to the right
= 0 symmetric, e.g. Gausian
< 0 tail to the left

$$\kappa_{\mathcal{N}} = \frac{\overline{\delta\mathcal{N}^4}}{(\overline{\delta\mathcal{N}^2})^2} = 3 + \frac{M_4}{S_0^2\,\overline{\mathcal{N}}}. = 3 + \frac{M_4}{S_0^3}$$

> 3 leptokurtic. Long tails
= 3 Gaussian
< 3 platykurtic. Short tails

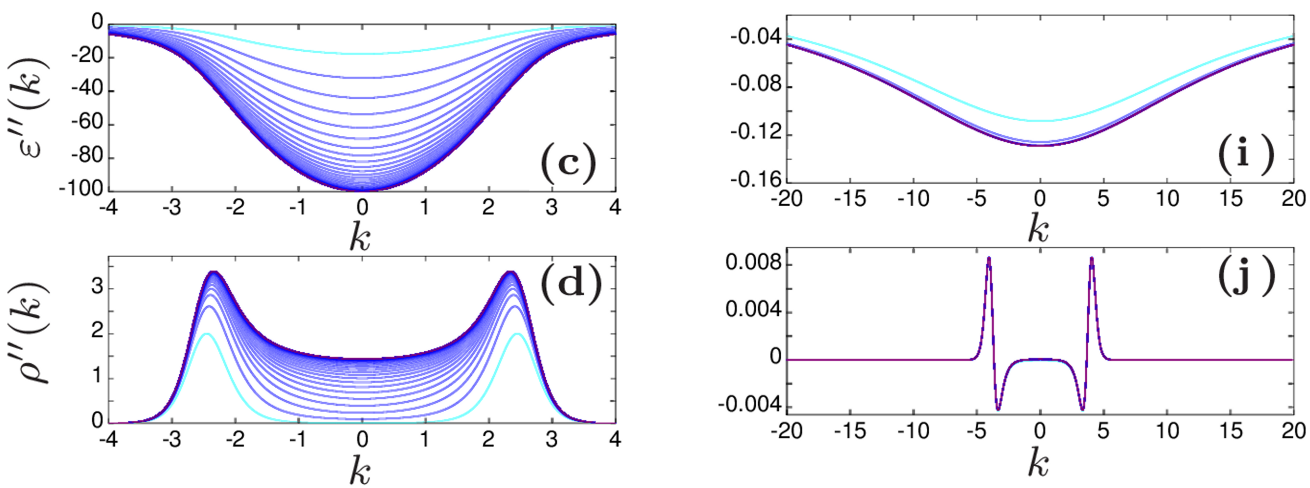
- Intensive quantity

$$M_3 = \frac{\langle \delta N^3 \rangle}{\langle \widehat{N} \rangle} = (k_B T)^2 \; \frac{\int_{-\infty}^{\infty} \rho''(k) dk}{\int_{-\infty}^{\infty} \rho(k) dk} \qquad \rho''(k) = \frac{\partial^2 \rho(k)}{\partial \mu^2}$$

- equations

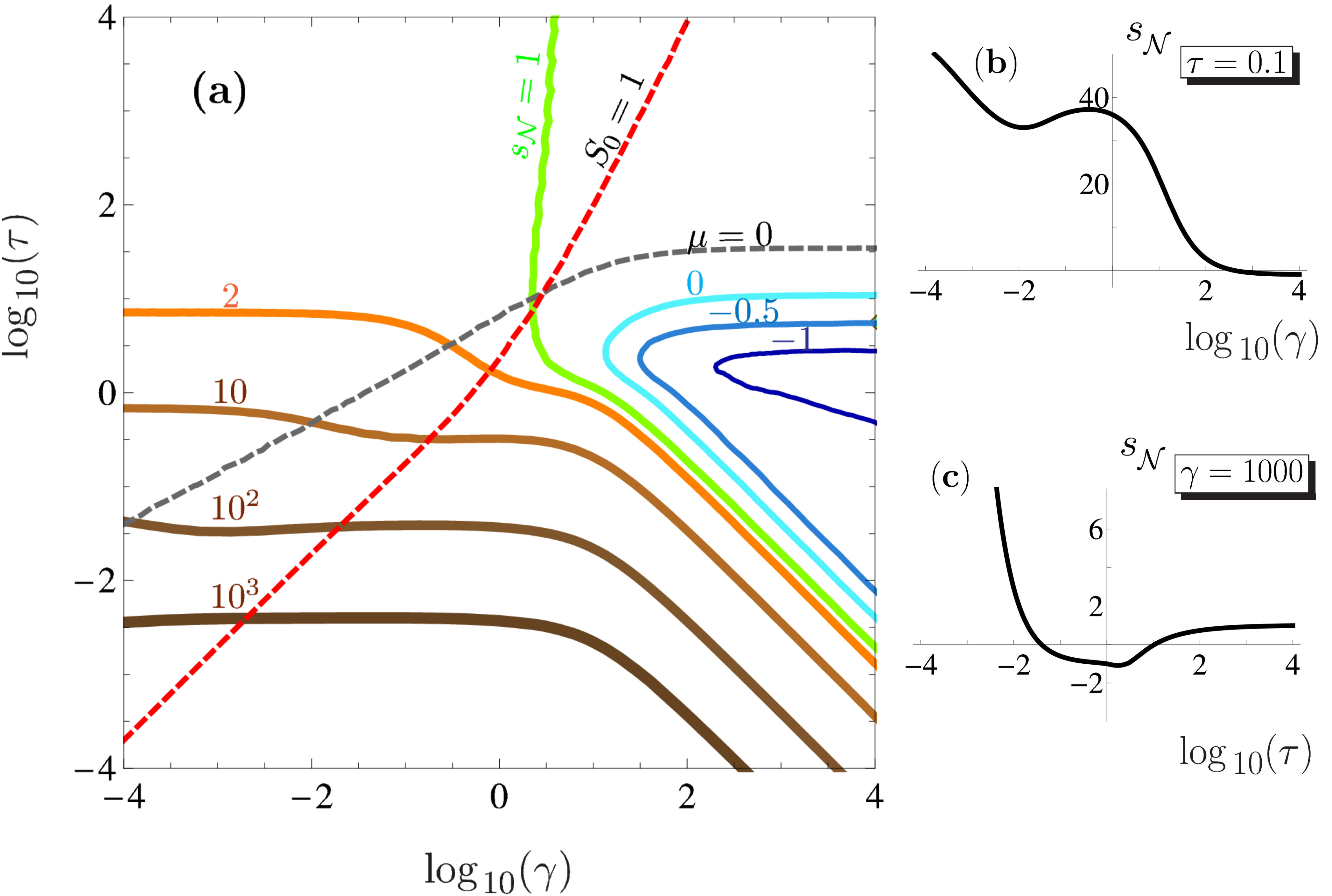
$$\left[2\rho'(k)\varepsilon'(k) + \rho(k)\varepsilon''(k) + \frac{\rho(k)\varepsilon'(k)^2}{T} \right] \frac{e^{\varepsilon(k)/T}}{T} + \rho''(k) \left[1 + e^{\varepsilon(k)/T} \right] = \frac{g}{\pi} \int_{-\infty}^{\infty} \frac{dq \; \rho''(q)}{g^2 + (k-q)^2}$$

$$\varepsilon''(k) = \frac{g}{\pi} \int_{-\infty}^{\infty} \frac{dq}{g^2 + (k-q)^2} \left\{ \frac{\varepsilon''(q)}{1 + e^{\varepsilon(q)/T}} - \frac{1}{T} \left(\frac{\varepsilon'(q)}{1 + e^{\varepsilon(q)/T}} \right)^2 \right\}$$

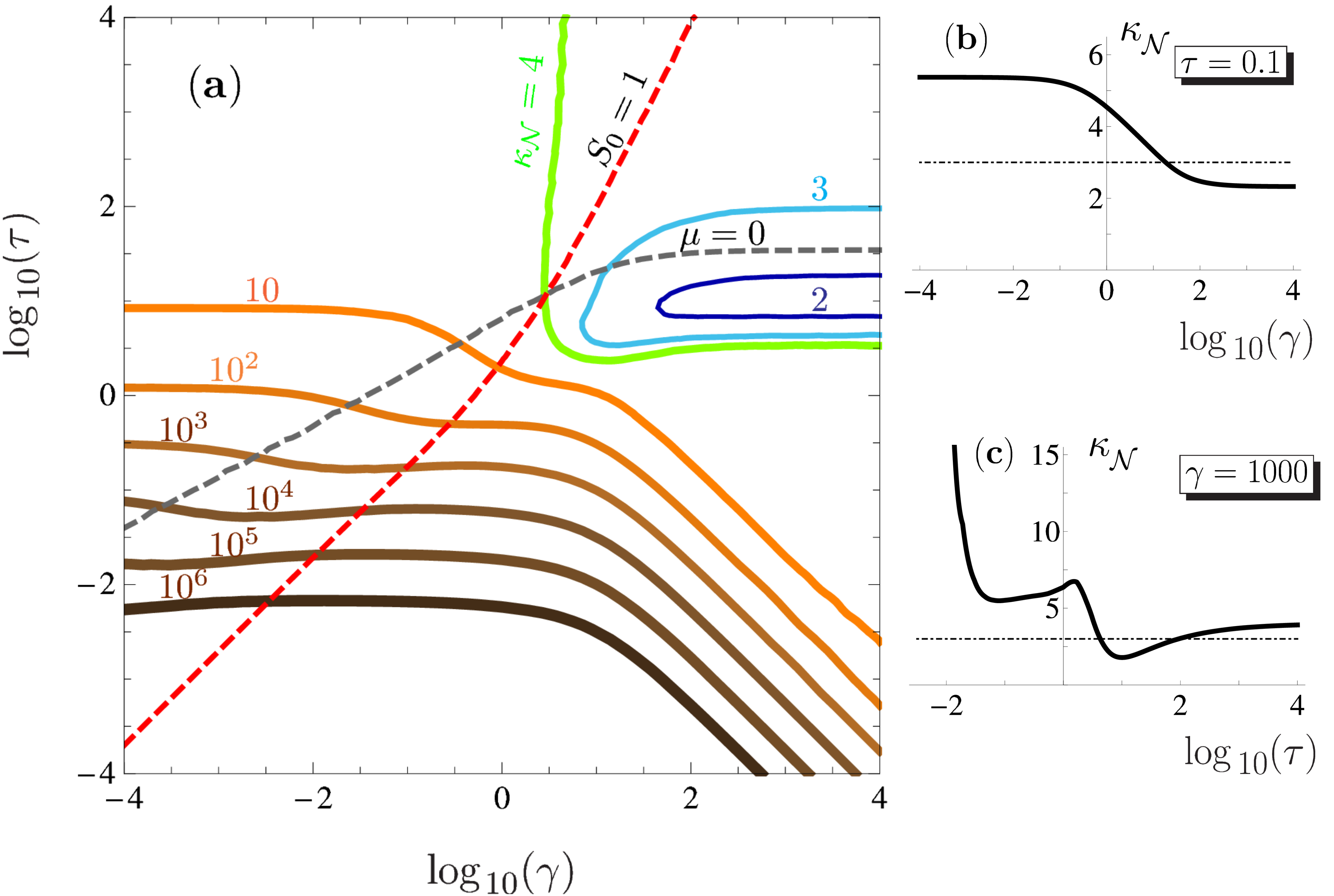


$$s_{\mathcal{N}} = \frac{M_3}{S_0^2}$$

Skewness of density grain distribution



Kurtosis of density grain distribution

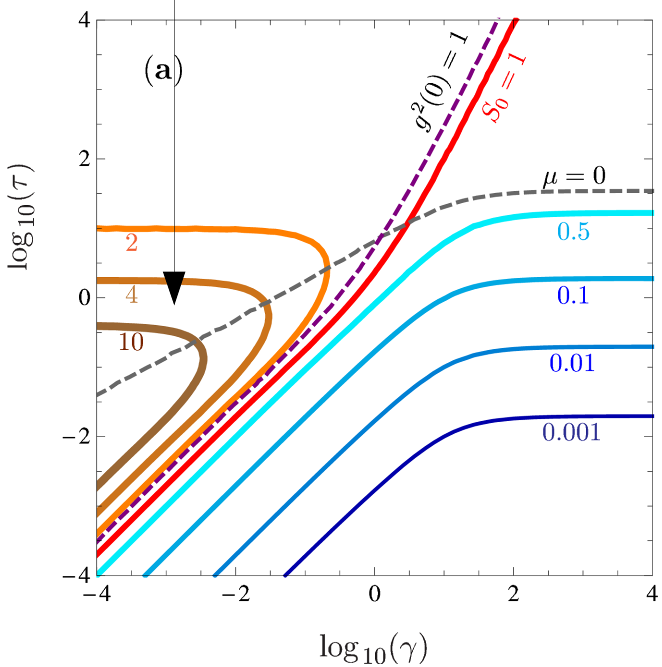


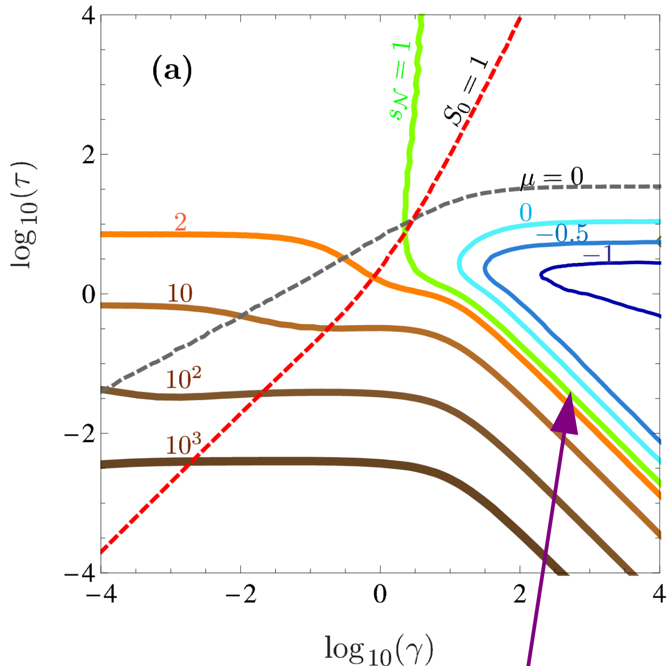
Distribution properties overall

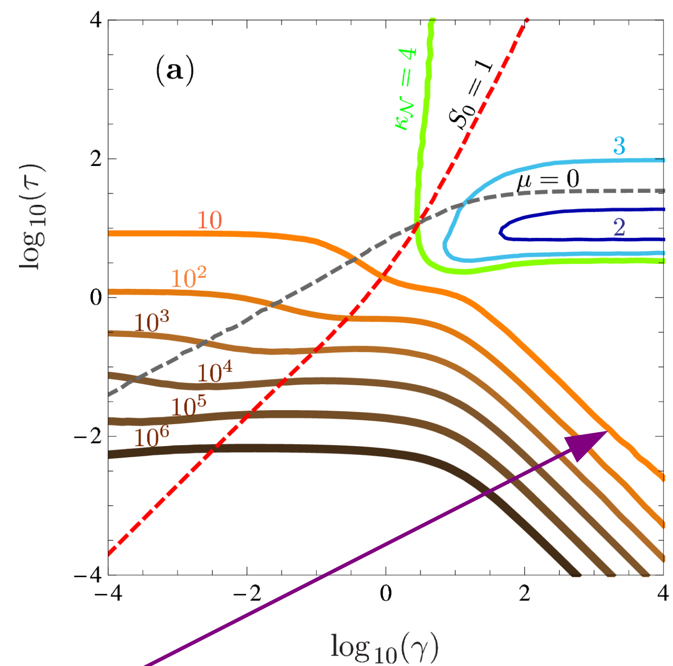
Quasicondensate:
large density grains
long distribution tails at large $\mathcal{N}$

Classical shot noise: $s_{\mathcal{N}} = \frac{1}{\sqrt{\overline{\mathcal{N}}}} \to 1$

$S_0 = \overline{\mathcal{N}} \to 1$ $\kappa_{\mathcal{N}} = 3 + \frac{1}{\overline{\mathcal{N}}} \to 4$







Unexpected crossover

(remains unexplained physically)

$$\tau \approx \frac{C}{\gamma}, \qquad \text{i.e.} \qquad k_B T \approx \frac{C}{2}\frac{\hbar^4 n^3}{m^2 g}$$

$$C \approx 30$$

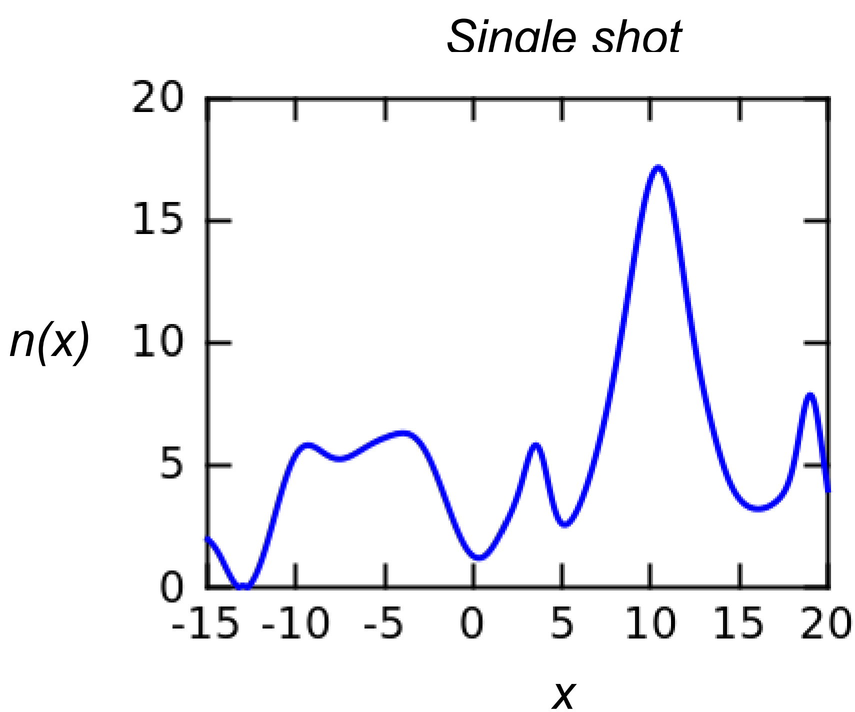
Nature of density grains

Question: Lumps seen in single experimental images

= ?

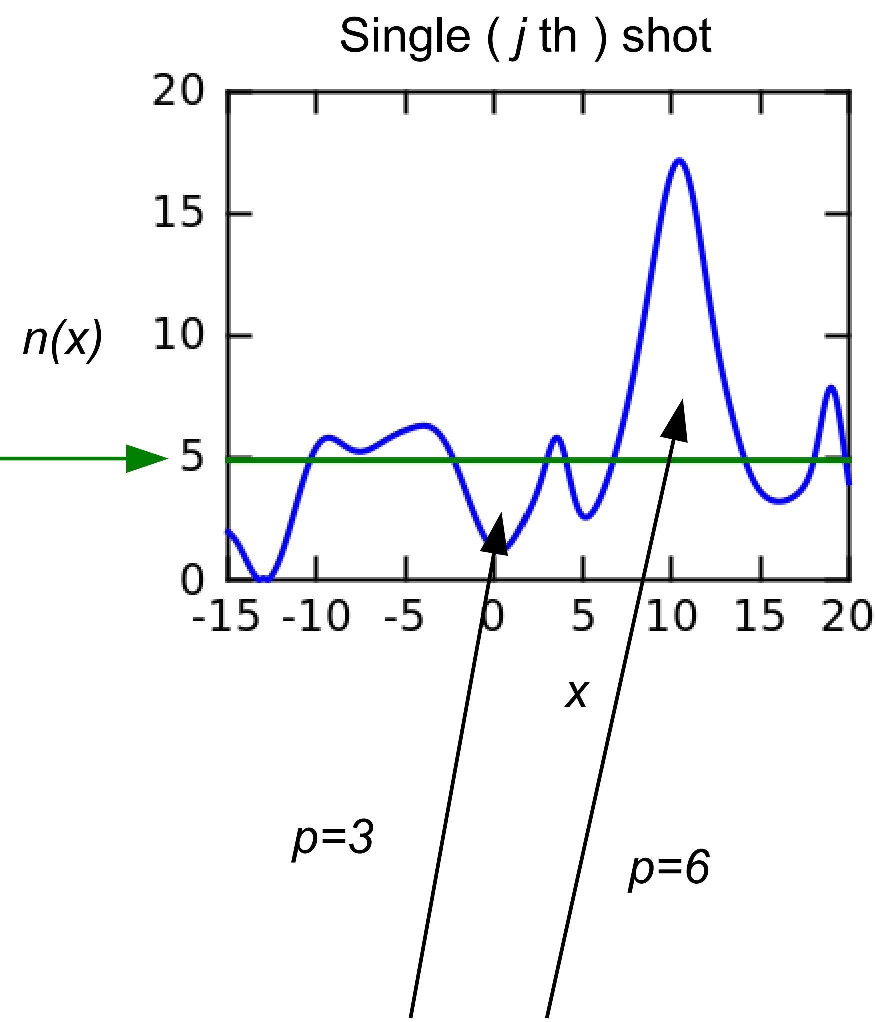
Independent density fluctuations

Is this "Lump conjecture" true?



Experimental test

$$\overline{n}_j = \frac{1}{l}\int_{-l/2}^{l/2} dx\, n_j(x)$$



Statistics of N_{jp} will be the same as

Statistics of $\mathcal{N}$

if/when the Lump conjecture is correct

$$N_{jp} = \int_{x_{(p)}}^{x_{(p+1)}} dx\; n_j(x)$$

Conclusions

- New set of equations to extract number fluctuations from Yang-Yang solution

  → New exact results for all T, g

- Statistics and size of density grains

  Large density grains in the quasicondensate

  Peculiar behavior and unknown crossover in the fermionized regime

  and so on ...

- Experimental proposal to test:

  Are *density lumps in single shots independent fluctuations?*

- Thank you also to:

  Karen Kheruntsyan *University of Queensland, Brisbane*

  Isabelle Bouchoule *Institute d'Optique, Palaiseau*

• Intensive quantities

• Measuring ion box l

•

$$S_0 = \frac{1}{n}\frac{\partial\, \mathrm{var} N^{(l)}}{\partial l},$$

$$M_3 = \frac{1}{n}\frac{\partial \langle (\delta N^{(l)})^3\rangle}{\partial l},$$

$$M_4 = \frac{1}{n}\frac{\partial}{\partial l}\left[\langle (\delta N^{(l)})^4\rangle - 3(\mathrm{var} N^{(l)})^2\right]$$

$$S_0 = \frac{\langle \delta \widehat{N}^2\rangle}{\langle \widehat{N}\rangle} = \frac{\overline{\delta \mathcal{N}^2}}{\overline{\mathcal{N}}},$$

$$M_3 = \frac{\langle \delta \widehat{N}^3\rangle}{\langle \widehat{N}\rangle} = \frac{\overline{\delta \mathcal{N}^3}}{\overline{\mathcal{N}}},$$

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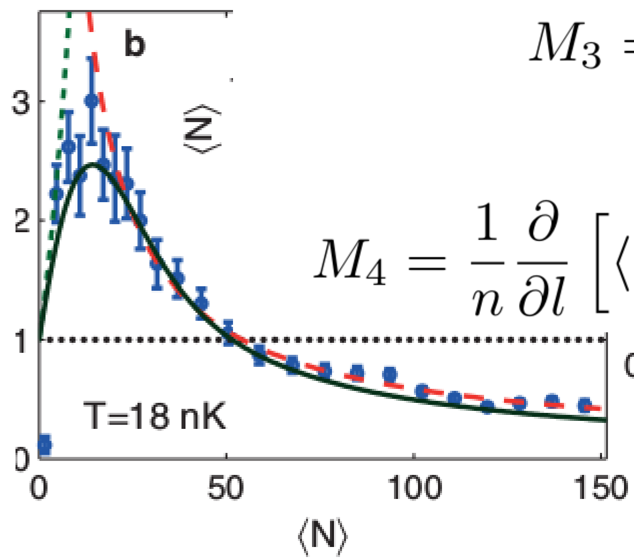
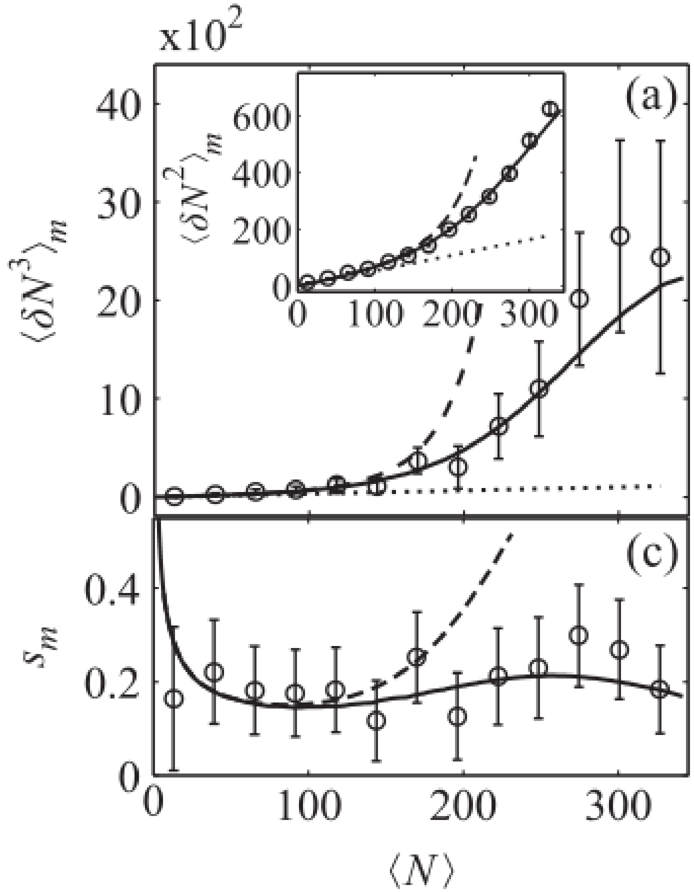


$$N_j^{(l)} = \int_{-l/2}^{l/2} dx\ n_j(x)$$



Armijo,Jacqmin, Kheruntsyan, Bouchoule, PRL ***105****, 230402 (2010)*