Stochastic Gauges

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Attempts to improve on the Positive P-distribution, and ideas on what you might be able to do with such improvements.

(Work in progress with P. Drummond)

Outline

- What's so interesting about P-distributions, and why look for improvements.

- A quick review of using P-distributions to look at the evolution of quantum states. (With example)

- Relevance to BEC's

- A variant P-distribution, and its application for

$$\widehat{H} = \widehat{a}^\dagger \widehat{a}^\dagger \widehat{a}\, \widehat{a}.$$

- Stochastic Gauges.

- *if time permits:* A P-like distribution in squeezed states.

Why?

- P-distributions (and similar distributions, e.g. Wigner) are widely used to look at quantum evolution of states via stochastic equations.

- Stochastic equation methods are the *ONLY* known practical way to do a full quantum treatment of many-many body systems. (e.g. BEC's).

- So... improving their behaviour can lead to solution of new problems.

- Hours of fun! ??

Many-Many Body Problems

It has been (and is) claimed (e.g. famously by Feynman) that full quantum evolution of systems involving a large number of bodies is impossible to model on classical computers.

The idea being that if you have N bodies, each with D energy levels (say), then Hilbert space has

$$D^N$$

dimensions.

e.g. for just 20 10-energy-level particles, that's

$$100,000,000,000,000,000,000$$

simultaneous differential equations to solve.

(piece of cake!)

But. . .

But... you can simulate the state evolution using stochastic equations. In many cases you only have

$$\text{some constant} \quad \times \quad N$$

stochastic equations!

E.g: Drummond and Corney treated the evaporative cooling of ions, and formation of a BEC using the positive P-distribution.

[P. D. Drummond and J. F. Corney,
Phys. Rev. A **60**, R2661 (1999)]

There were 10,000 atoms!

Clearly stochastic methods are useful here!

A quick review, or
"How does it all work?"

Lets consider a squeezing hamiltonian as an example:

$$\widehat{H} = i\hbar \left[\widehat{a}^{\dagger 2} - \widehat{a}^2\right]$$

So the *Master equation* (no damping) is

$$\dot{\rho} = \widehat{a}^{\dagger 2}\rho - \widehat{a}^2\rho - \rho\widehat{a}^{\dagger 2} + \rho\widehat{a}^2$$

We can write

$$\hat{\rho} = \int P(\alpha, \beta) \frac{||\alpha >< \beta^*||}{< \beta^*||\alpha >} \; d^2\alpha \;\; d^2\beta$$

Where

$$||\alpha >= e^{\alpha \hat{a}^\dagger}|0>$$

are unnormalised (Bargmann) Coherent states of complex amplitude α.

and

$$P(\alpha, \beta)$$

is the *positive P-distribution* for a state ρ.

So, for example a Coherent state $|\alpha_o>$ has a positive P-distribution of

$$P(\alpha, \beta) = \delta(\alpha - \alpha_o)\delta(\beta - \alpha_o^*)$$

The nice thing about the states $||\alpha>$ is that they obey relations like:

$$\hat{a}||\alpha>=\alpha||\alpha>$$

which leads to

$$\hat{a}^2||\alpha>=\alpha^2||\alpha>$$

and so,

$$\hat{a}^2\hat{\rho}=\int \left[\quad \alpha^2 \quad P(\alpha,\beta) \quad \right] \frac{||\alpha><\beta^*||}{<\beta^*||\alpha>} d^2\alpha \quad d^2\beta$$

compare this to

$$\hat{\rho}=\int \quad [\quad P(\alpha,\beta) \quad] \quad \frac{||\alpha><\beta^*||}{<\beta^*||\alpha>} d^2\alpha \quad d^2\beta$$

You can see the correspondences that

$$\hat{\rho} \leftrightarrow P(\alpha,\beta)$$

$$\hat{a}^2\hat{\rho} \leftrightarrow \alpha^2 P(\alpha,\beta)$$

You can obtain similar such relations for the other terms in the Master equation. For example:

$$\hat{a}^{\dagger}||\alpha >= \frac{\partial}{\partial \alpha}||\alpha >$$

which leads to

$$\hat{a}^{\dagger 2}||\alpha >= \frac{\partial^2}{\partial \alpha^2}||\alpha >$$

Which after some calculus, leads to

$$\hat{a}^{\dagger 2}\hat{\rho} \leftrightarrow \left[\beta^2 - \frac{\partial}{\partial \alpha}2\beta + \frac{\partial^2}{\partial \alpha^2}\right] P(\alpha, \beta)$$

In any case, making the correspondence for the whole Master equation:

$$\text{Master Equation} \leftrightarrow$$

$$\frac{\partial P}{\partial t} = \left[-2\frac{\partial}{\partial \alpha}\beta - 2\frac{\partial}{\partial \beta}\alpha + \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} \right] P$$

Which is a *Fokker-Planck Equation* for the (positive) distribution function $P(\alpha, \beta)$ over two complex variables.

By standard methods this can be rewritten in equivalent form as a set of two complex stochastic equations.

The *first* derivative terms lead to deterministic terms in these equations,

while the *second* derivative terms lead to noise terms in those equations.

For the example discussed, the stochastic equations are just

$$\dot{\alpha} = 2\beta + \xi(t)$$

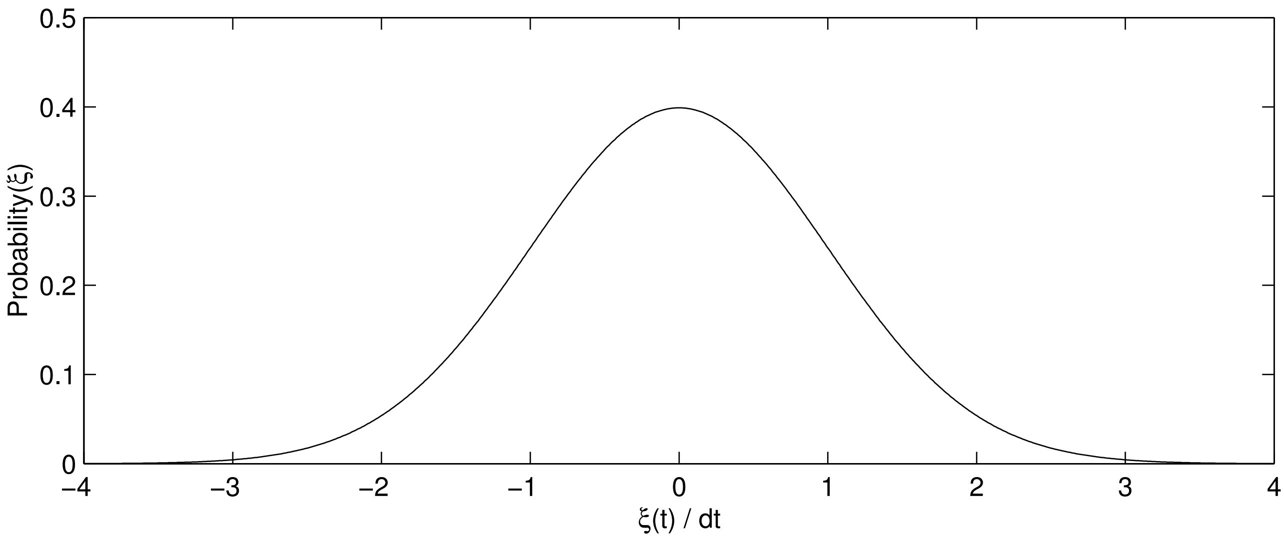
$$\dot{\beta} = 2\alpha + \xi'(t)$$

Where the ξ and ξ' are independent real gaussian noises of mean zero and variance dt. i.e.

$$< \xi(t) >= 0$$

$$< \xi(t)\xi(t) >=< \xi'(t)\xi'(t) >= dt$$

$$< \xi(t)\xi'(t) >= 0$$



So what can you do with these equations?

$$\dot{\alpha} = 2\beta + \xi(t)$$

$$\dot{\beta} = 2\alpha + \xi'(t)$$

One finds that to calculate expectation values of moments, you need only take averages over the complex variables α and β. For example:

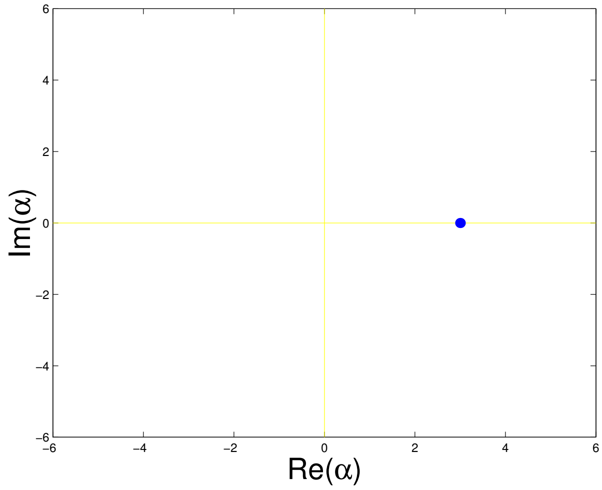
$$< \hat{a}^\dagger \hat{a} > = < \alpha\beta >$$

$$< \hat{a} + a^\dagger > = < \alpha + \beta >$$

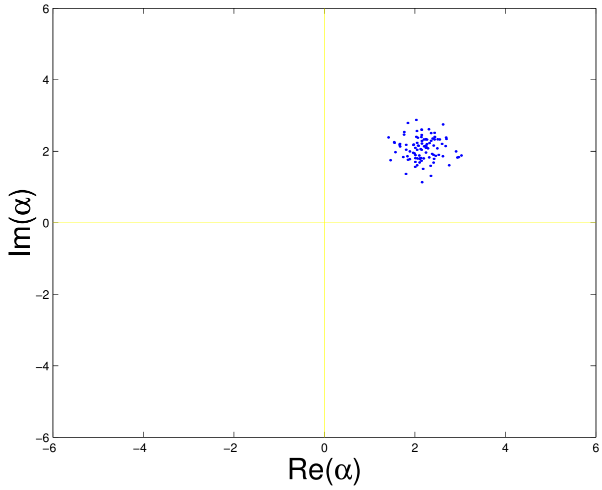
$$-i < \hat{a} - a^\dagger > = -i < \alpha - \beta >$$

So you do N (e.g. 100) runs of the stochastic differential equations, starting with the initial conditions distributed according to $P(\alpha, \beta, t = 0)$.

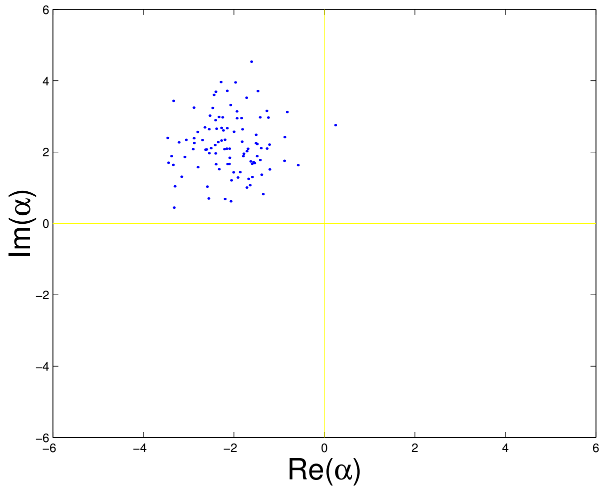
Then to get (say) the expectation value of the x quadrature $< \hat{a} + \hat{a}^\dagger >$, you average $< \alpha + \beta >$ over your N runs.



Im(α)
Re(α)

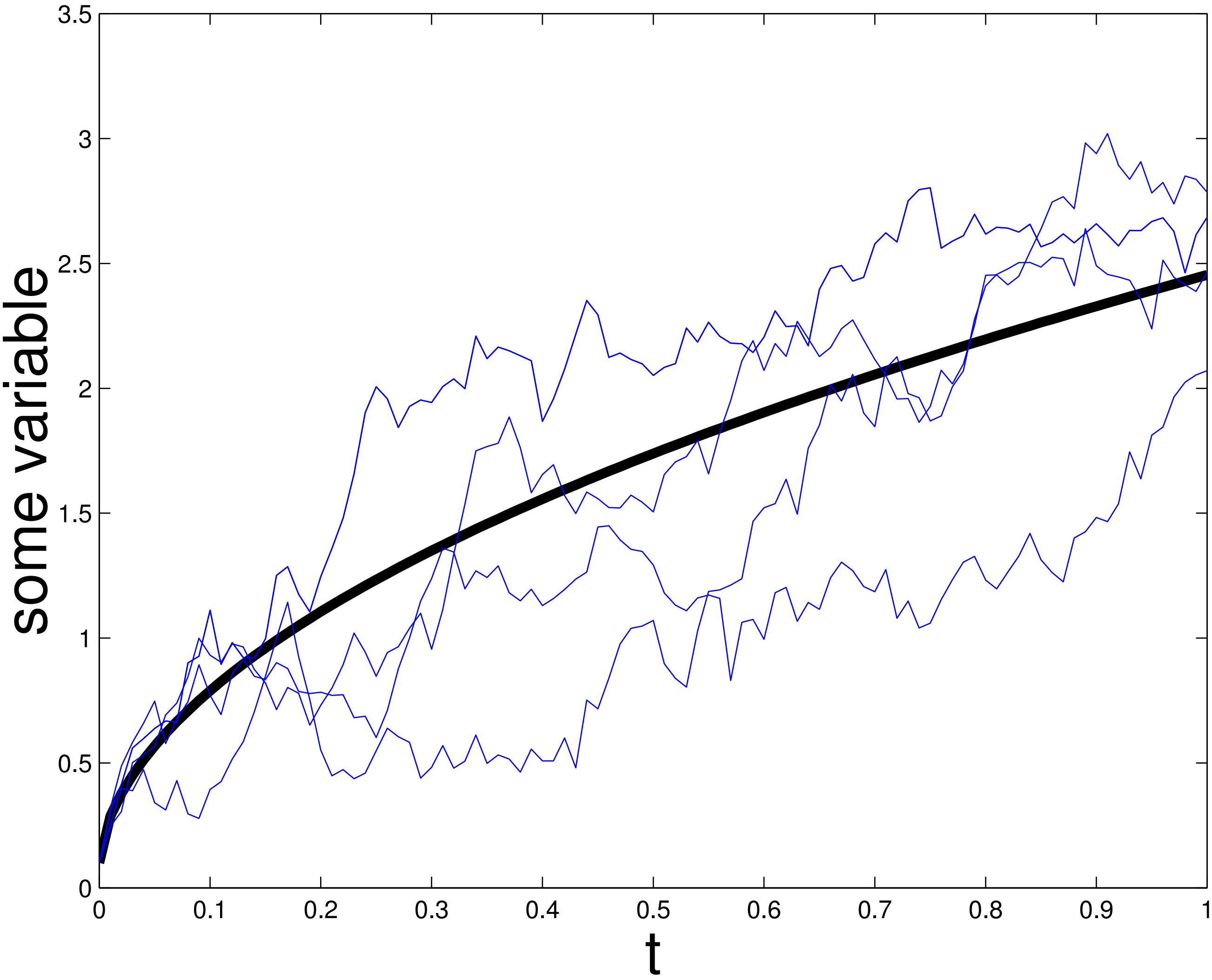


Im(α)
Re(α)

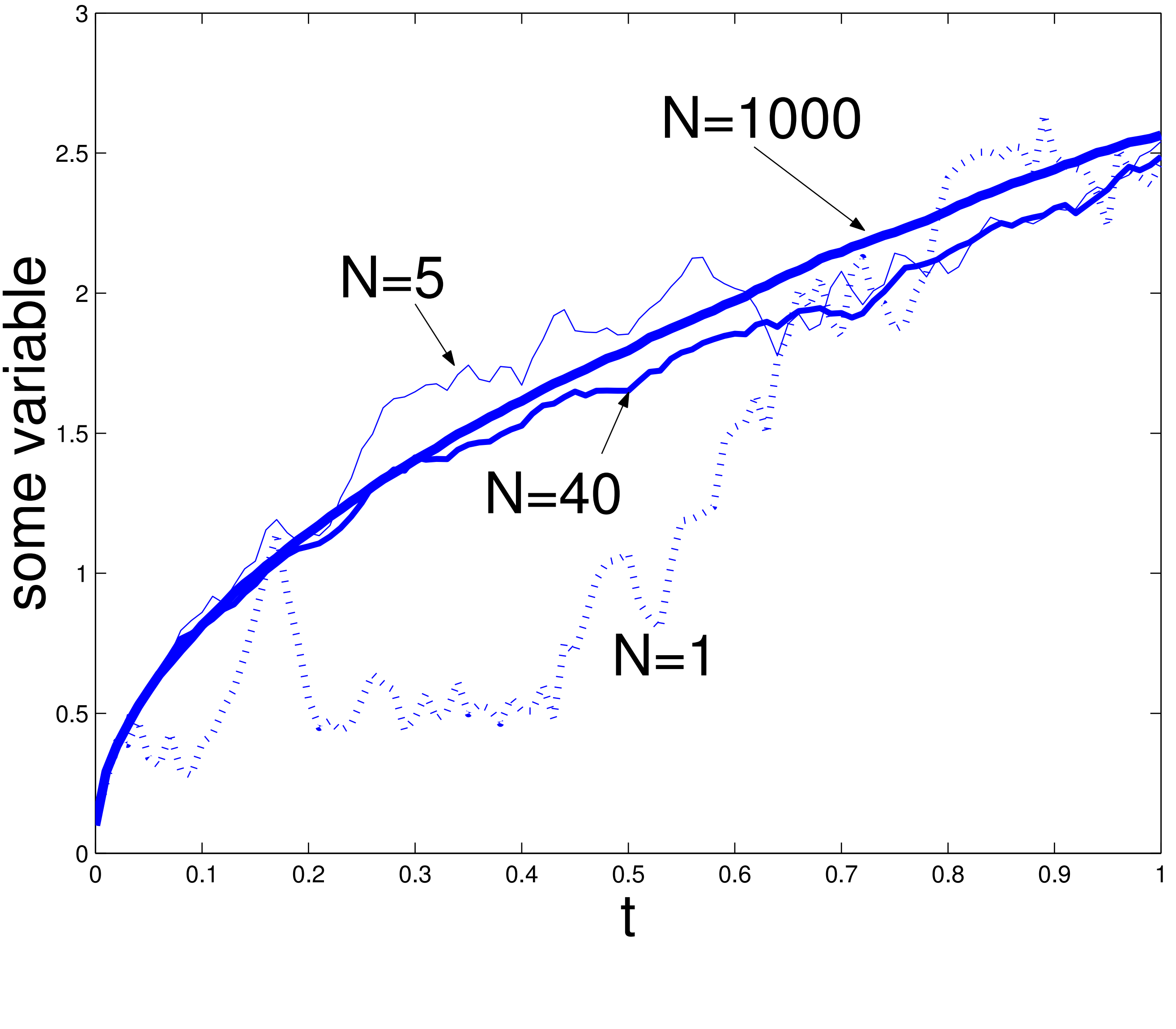


Im(α)
Re(α)

Individual paths, and the mean:



Averages over different numbers of paths:



Applications to BEC's, and problems

A good model for a single-species neutral atom BEC in a trap has

$$\widehat{H} = \int d^3\mathbf{x}\Big[\frac{\hbar^2}{2m}\nabla\widehat{\Psi}^\dagger(\mathbf{x})\nabla\widehat{\Psi}(\mathbf{x}) + V(\mathbf{x})\widehat{\Psi}^\dagger(\mathbf{x})\widehat{\Psi}(\mathbf{x})$$
$$+\widehat{\Psi}^\dagger(\mathbf{x})\widehat{R}(\mathbf{x}) + \widehat{\Psi}(\mathbf{x})\widehat{R}^\dagger(\mathbf{x})$$
$$+\frac{1}{2}\int d^3\mathbf{y}U(\mathbf{x}-\mathbf{y})\widehat{\Psi}^\dagger(\mathbf{x})\widehat{\Psi}^\dagger(\mathbf{y})\widehat{\Psi}(\mathbf{x})\widehat{\Psi}(\mathbf{y})\Big]$$

with $\widehat{\Psi}(\mathbf{x})$ boson operators at position $\mathbf{x}$.

Upon conversion to free-field modes, you get the following sort of terms in the hamiltonian:

$$\widehat{a}^\dagger\widehat{a}$$

absorption

$$\widehat{a}^{\dagger 2}\widehat{a}^2$$

The first two terms lead only to drift in the stochastic equations for the positive P-distribution, but unfortunately the last (quartic) term is not as stable as one would hope.

Consider what happens for the one-mode Hamiltonian. . .

$$\widehat{H} = \hbar \widehat{a}^{\dagger 2} \widehat{a}^2$$

The stochastic equations are

$$\dot{\alpha} = -i\alpha(1 + 2\alpha\beta) + (1 - i)\alpha\xi(t)$$

$$\dot{\beta} = i\beta(1 + 2\alpha\beta) + (1 + i)\beta\xi'(t)$$

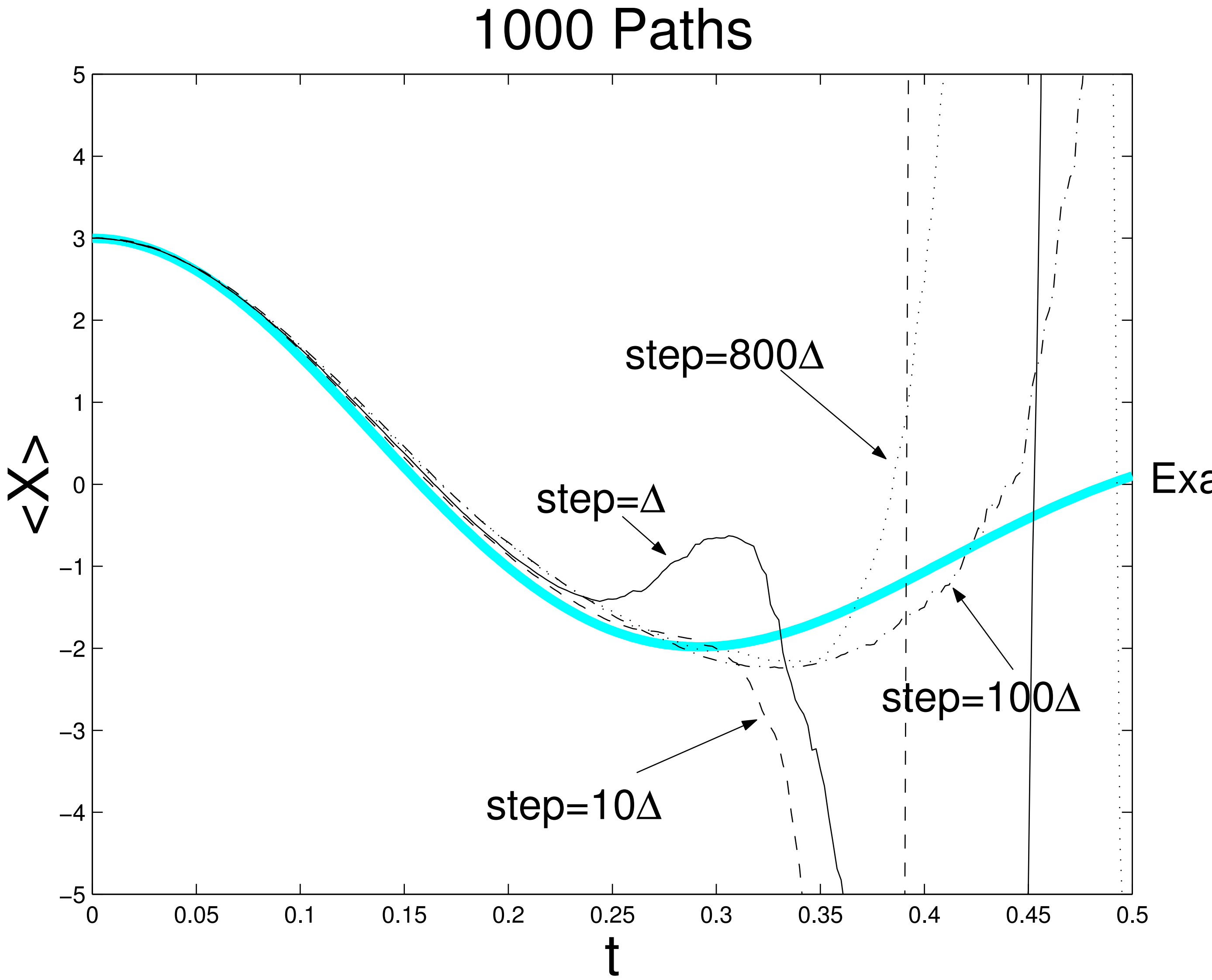
Where the variables and noises are the same as in the example:

α and β correspond to the coherent amplitude, and its conjugate (at least initially).

$\xi(t)$ and $\xi'(t)$ are gaussian noises of variance dt.

Note that the noise is multiplicative!

For the single-mode case, you can solve for the quadrature $< x >$ exactly.



The positive P-distribution does well up to a certain time, but unfortunately after about $t \approx 0.3$, the $< x >$ errors are resistant to step size.

This problem comes about because around this time, the sampling error (i.e. the *spread* of the distribution) increases dramatically due to instabilities in the stochastic equations.

But there is hope. . .

Perhaps we can expand $\hat{\rho}$ in terms of a different family of states, than $|\alpha>$.

The Positive P-distribution expands in terms of the kernel

$$\Lambda = |\alpha><\beta^*|$$

where

$$\hat{\rho} = \int P(\alpha, \beta) \left(\frac{\Lambda}{\mathrm{Tr}[\Lambda]} \right) d^2\alpha d^2\beta$$

This is identical *in form* to a mixture of states Λ, however, the Λ are neither Hermitian nor positive.

A Conjecture:

Perhaps it might be better to narrow down the possible kernels Λ, and force them to be Hermitian?

Proposed Hermitian Kernel

$$\Lambda_H = e^{i\theta}|\alpha >< \beta^*| + e^{-i\theta}|\beta >< \alpha^*|$$

$$\hat{\rho} = \int P(\alpha, \beta, \theta) \left(\frac{\Lambda_H}{\mathrm{Tr}[\Lambda_H]} \right) \, d^2\alpha \; d^2\beta \; d\theta$$

Now we have five real variables.

Basic Identities

Hermitian kernel:

$$\hat{a}\Lambda_H = \left[\frac{\alpha + \beta^*}{2} - i\left(\frac{\alpha - \beta^*}{2}\right)\frac{\partial}{\partial\theta}\right]\Lambda_H$$

$$\hat{a}^\dagger\Lambda_H = \left[\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta^*}\right]\Lambda_H$$

Positive-P kernel:

$$\hat{a}\Lambda = \alpha\Lambda$$

$$\hat{a}^\dagger\Lambda = \frac{\partial}{\partial\alpha}\Lambda$$

Basic Correspondences

Hermitian kernel:

$$\widehat{a}^\dagger \widehat{a} \widehat{\rho} \leftrightarrow \left[\mathrm{Re}[\alpha\beta] - T \ \mathrm{Im}[\alpha\beta] - \frac{\partial}{\partial \alpha}\alpha - \frac{\partial}{\partial \beta^*}\beta^* \right] P$$

$$\widehat{a} \widehat{\rho} \leftrightarrow \left[\frac{\alpha + \beta^*}{2} + iT\frac{\alpha - \beta^*}{2} + \frac{i}{2}\frac{\partial}{\partial \theta}(\alpha - \beta^*) \right] P$$

where

$$T = \tan\left(\theta + \mathrm{Im}[\alpha\beta^*]\right)$$

Positive-P kernel:

$$\widehat{a}^\dagger \widehat{a} \widehat{\rho} \leftrightarrow \left[\alpha\beta - \frac{\partial}{\partial \alpha}\alpha \right] P$$

$$\widehat{a}^\dagger \widehat{\rho} \leftrightarrow \frac{\partial}{\partial \alpha} P$$

Stochastic Gauge

In the positive P-representation there were only four real variables. (And the Glauber P-representation has only two, for example), but the Hermitian distribution has an extra real variable.

This gives a degree of freedom.

One finds the identities:

$$\left[1 + \frac{\partial^2}{\partial\theta^2}\right]\Lambda_H = 0$$

$$\left[\frac{\partial^2}{\partial\theta\partial\alpha} - i\frac{\partial}{\partial\alpha}\right]\Lambda_H = 0$$

$$\left[\frac{\partial^2}{\partial\theta\partial\beta} - i\frac{\partial}{\partial\beta}\right]\Lambda_H = 0$$

So these identities multiplied by

Any ARBITRARY function

must also equal zero!

This means that (taking the first identity)

$$0 \leftrightarrow F(\alpha, \alpha^*, \beta, \beta^*, \theta) \left[\frac{\partial^2}{\partial \theta^2} + 2\frac{\partial}{\partial \theta} T \right] P$$

For **ANY** arbitrary function F. And since it corresponds to zero, it can be added at will to the Fokker- Planck Equation.

Choosing F is similar to choosing a gauge in field theory, in that it is a function which has no effect on physical quantities, but can be chosen at will to make calculations more convenient.

Clearly, Any identity of the form

$$\text{[some operator]}\ \Lambda = 0$$

gives rise to such a stochastic gauge.

For this Hamiltonian, and kernel Λ_H we have three such.

For calculations (due to convenience) we have chosen somewhat different (real) variables, which give the following stochastic equations.

$$\begin{aligned}
\dot{x} &= \frac{1}{2}\left[(n_1 + n_2) + T(n_1 - n_2) + 2FT\right] + \xi \\
\dot{\bar{x}} &= \frac{1}{2}\left[(n_1 - n_2 - T(n_1 + n_2) - 2\bar{F}T\right] + \bar{\xi} \\
\dot{y} &= F \\
\dot{\bar{y}} &= \bar{F} \\
\dot{\theta} &= -F\left\{\frac{1}{2}\left[n_1 + n_2 + T(n_1 - n_2) + 2TF\right] + F\xi\right\} \\
&\quad +\bar{F}\left\{\frac{1}{2}\left[n_1 - n_2 - T(n_1 + n_2) - 2T\bar{F}\right] + \bar{F}\bar{\xi}\right\}
\end{aligned}$$

Note how all the terms in $\dot{\theta}$ have F or $\bar{F}$ factors!

where

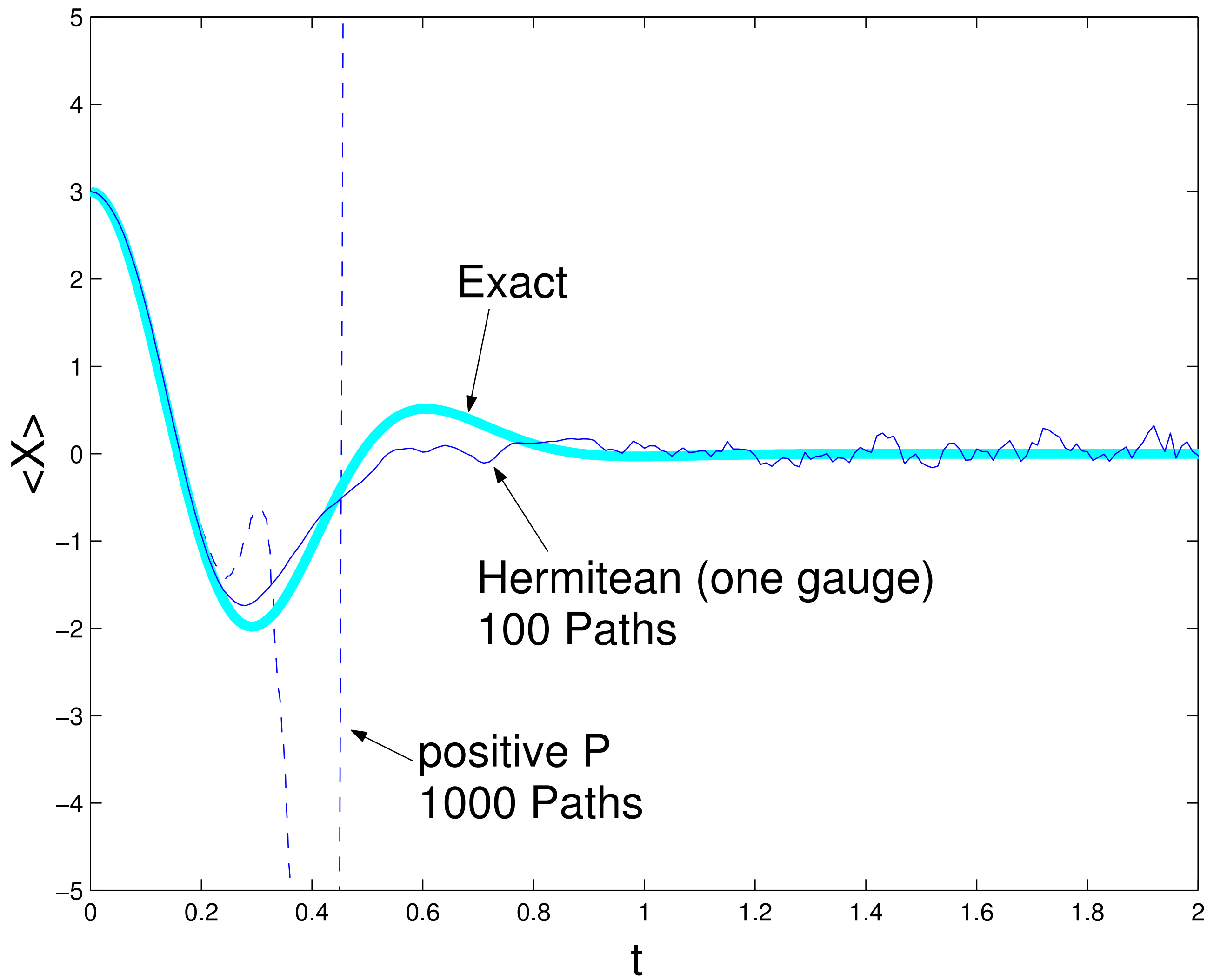
$$\begin{aligned}
n_1 &= \exp\left[(x + \bar{x} + y + \bar{y})/2\right] \cos\left[(y - \bar{y} - x + \bar{x})/2\right] \\
n_2 &= \exp\left[(x + \bar{x} + y + \bar{y})/2\right] \sin\left[(y - \bar{y} - x + \bar{x})/2\right] \\
T &= \tan(\theta + n_2)
\end{aligned}$$

From preliminary calculations, a promising choice of gauge seems to be

$$F = \bar{F} = -\frac{1}{2}(n_1 - Tn_2) = -\frac{< \hat{a}^\dagger \hat{a} >}{2} = -\frac{\bar{n}}{2}$$

which gives equations

$$\begin{aligned}
\dot{x} &= \frac{1}{2}\left[\bar{n} + n_2(1+T^2)\right] + \xi \\
\dot{\bar{x}} &= \frac{1}{2}\left[\bar{n} - n_2(1+T^2)\right] + \bar{\xi} \\
\dot{y} &= -\frac{1}{2}\bar{n} \\
\dot{\bar{y}} &= -\frac{1}{2}\bar{n} \\
\dot{\theta} &= \frac{1}{2}\bar{n}n_2(1+T^2) + \frac{\bar{n}}{2}(\xi - \bar{\xi})
\end{aligned}$$


Exact
Hermitean (one gauge)
100 Paths
positive P
1000 Paths
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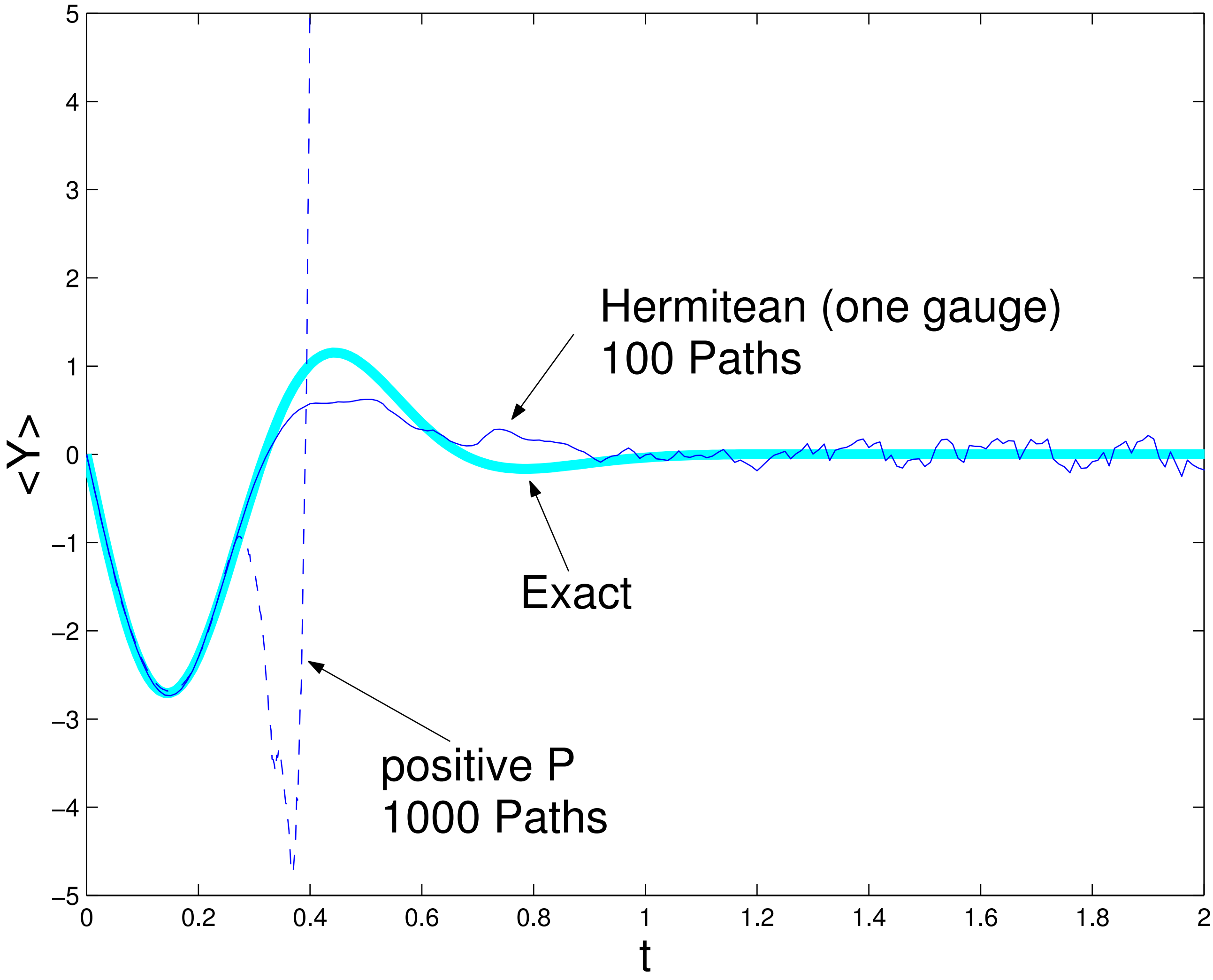

Hermitean (one gauge)
100 Paths
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This research is ongoing at the moment. We'll see what we get!

One of the he ultimate aims is to make first-principles calculations to investigate the behaviour of a BEC after it has condensed, but these methods may have broader applications.

Apart from making calculation easier, new types of P-like distributions may allow simulations of Hamiltonians which cannot be treated using the usual P-distributions.

We have investigated a P-like distribution in squeezed states. This one has the kernel

$$\Lambda_S = |\alpha, \zeta >< \bar{\alpha}^*, \bar{\zeta}^*|$$

where $|\alpha, \zeta >$ is a squeezed state, having mean quadratures the same as the coherent state $|\alpha >$.

This distribution allows both the α's and ζ's to vary.

Features seen include:

- Squeezing Hamiltonian produces only drift in the DE's. This allows exact solutions for the evolution of arbitrary squeezed states under arbitrary damped, multimode pumped squeezing.

- Terms like $\hat{a}^3$, $\hat{a}^{\dagger 3}\hat{a}$ or $\hat{a}^4$ in the Hamiltonian can be treated fully, whereas previous distributions always produced third order derivatives in the FPE.

Exact Two-mode Squeezing

$$\hat{H} = i\hbar[\hat{a}^\dagger \hat{b}^\dagger - \hat{a}\hat{b}]$$

Stochastic equations:

$$\begin{aligned}
\dot{\alpha}_1 &= -\alpha_1\beta_{12} - \alpha_2\beta_1 \\
\dot{\alpha}_2 &= -\alpha_2\beta_{12} - \alpha_1\beta_2 \\
\dot{\beta}_1 &= -2\beta_1\beta_{12} \\
\dot{\beta}_2 &= -2\beta_2\beta_{12} \\
\dot{\beta}_{12} &= 1 - \beta_{12}^2 - \beta_1\beta_2
\end{aligned}$$

α_i : coherent amplitude of mode i

β_i : squeezing of mode i

β_{12} : two-mode squeezing

N.B. if

$$|z, \xi >= e^{z\hat{a}^\dagger - z^*\hat{a}} e^{\xi\hat{a}^{\dagger 2}/2 - \xi^*\hat{a}^2/2}|0>$$

$$\alpha = z - z^*\beta$$

$$\beta = \arg(\xi)\tanh|\xi|$$

Solutions for squeezing:

$$\beta_i(t) = \frac{2\beta_i(0)}{D(t)}$$

$$\begin{aligned} \beta_{12}(t) &= \frac{2\beta_{12}(0)}{D(t)} \cosh(2t) \\ &\quad + \frac{(1 - \det[B(0)])}{D(t)} \sinh(2t) \end{aligned}$$

$$\begin{aligned} D(t) &= 1 + \det[B(0)] + (1 - \det[B(0)]) \cosh(2t) \\ &\quad + 2\beta_{12}(0) \sinh(2t) \end{aligned}$$

$$B(0) = \begin{bmatrix} \beta_1(0) & \beta_{12}(0) \\ \beta_{12}(0) & \beta_2(0) \end{bmatrix}$$

Thank You