Separability Criteria

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If you have a density matrix, how do you tell whether it is separable?

- Will mostly give an overview of recent developments by various people regarding how to tell a mixed separable state from an entangled one. Particularly the *Partial transposition condition*.

- Will mention some work that has been carried out in this field last year (1999) at U. Q. by

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  and at the University of New Mexico by

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  Regarding the separability of noisy states.

A composite state $\widehat{\rho}$ *of two subsystems* A *and* B *is separable if it can be written*

$$\widehat{\rho} = \sum_i p_i \left(\widehat{\rho}_i^A \otimes \widehat{\rho}_i^B \right)$$

with $p_i > 0$

Separable = Cannot be produced by using only operations local to one subsystem, even if they are coordinated between the subsystems.

For pure states it's easy. If it's separable,

$$|\Psi\rangle = \mathsf{Tr}_B\left[\widehat{\rho}\right] \otimes \mathsf{Tr}_A\left[\widehat{\rho}\right] = |\psi\rangle_A \otimes |\psi\rangle_B$$

So just take the partial traces of $\widehat{\rho}$ and see if they're pure.

But for mixed states, it's not so easy.

An example: Consider a (2×2) system which is a mixture of a maximally entangled (singlet) state $|E\rangle$ and maximally mixed state $\widehat{M}$.

$$\widehat{\rho}(\varepsilon) = (1-\varepsilon)\widehat{M} + \varepsilon \, |E\rangle \, \langle E|$$

$$\widehat{M} = \frac{1}{4} \left[|11\rangle \, \langle 11| + |12\rangle \, \langle 12| + |21\rangle \, \langle 21| + |22\rangle \, \langle 22| \right]$$

$$|E\rangle = \frac{1}{\sqrt{2}} \left[|11\rangle + |22\rangle \right]$$

It is entangled when

$$\varepsilon > \frac{1}{3}$$

but separable for smaller values, even though it can never be written as a product state.

Really? What does the ensemble look like for $\varepsilon = 1/3$?

Define the density matrices

$$P_{\pm i} = \frac{1}{2}(I \pm \sigma_i)$$

then

$$P_{+1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad P_{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$P_{+2} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \qquad P_{-2} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$P_{+3} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad P_{-3} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

And

$$\hat{\rho}(1/3) = \frac{1}{6}\sum_{i=1}^{3}\left[P_{+i} \otimes P_{+i} + P_{-i} \otimes P_{-i}\right]$$

Partial Transposition Criterion

[A. Peres, Phys. Rev. Lett. **77**, 1413 (1996)]

A simple algebraic condition, which is *necessary* for $\widehat{\rho}$ to be separable.

If $\widehat{\rho}$ is separable, then its *partial transposition* ρ^{T_A} with respect to one subsystem A has positive eigenvalues.

Conversely, if ρ^{T_A} has negative eigenvalues then $\widehat{\rho}$ is entangled.

What does "partial transposition" mean?

write $\widehat{\rho}$ out in terms of matrix elements:

$$\widehat{\rho} = \sum_{ijmn} \rho_{ij,mn} |im\rangle \langle jn|$$

then

$$\rho^{T_A} = \sum_{ijmn} \rho_{ji,mn} |im\rangle \langle jn|$$

Example: the noisy singlet state from before

$$\hat{\rho}(\varepsilon) = (1-\varepsilon)\frac{\hat{I}}{4} + \varepsilon \left|E\right\rangle \left\langle E\right|$$

matrix form:

$$\hat{\rho}(\varepsilon) = \frac{1}{4} \begin{pmatrix} 1+\varepsilon & 0 & 0 & 2\varepsilon \\ 0 & 1-\varepsilon & 0 & 0 \\ 0 & 0 & 1-\varepsilon & 0 \\ 2\varepsilon & 0 & 0 & 1+\varepsilon \end{pmatrix} \begin{matrix} {\scriptstyle 11} \\ {\scriptstyle 12} \\ {\scriptstyle 21} \\ {\scriptstyle 22} \end{matrix}$$

$$\begin{matrix} {\scriptstyle 11} & {\scriptstyle 12} & {\scriptstyle 21} & {\scriptstyle 22} \end{matrix}$$

has eigenvalues

$$\frac{1+3\varepsilon}{4}\,, \quad \frac{1-\varepsilon}{4}\,, \quad \frac{1-\varepsilon}{4}\,, \quad \frac{1-\varepsilon}{4}$$

Whereas

$$\hat{\rho}^{T_A}(\varepsilon) = \frac{1}{4} \begin{pmatrix} 1+\varepsilon & 0 & 0 & 0 \\ 0 & 1-\varepsilon & 2\varepsilon & 0 \\ 0 & 2\varepsilon & 1-\varepsilon & 0 \\ 0 & 0 & 0 & 1+\varepsilon \end{pmatrix} \begin{matrix} {\scriptstyle 11} \\ {\scriptstyle 12} \\ {\scriptstyle 21} \\ {\scriptstyle 22} \end{matrix}$$

$$\begin{matrix} {\scriptstyle 11} & {\scriptstyle 12} & {\scriptstyle 21} & {\scriptstyle 22} \end{matrix}$$

has eigenvalues

$$\frac{1+\varepsilon}{4}\,, \quad \frac{1+\varepsilon}{4}\,, \quad \frac{1+\varepsilon}{4}\,, \quad \frac{1-3\varepsilon}{4}$$

Unfortunately this is not always a *sufficient* condition for separability. There may be entangled states which have positive partial transposition.

It is only sufficient for 2×2 and 2×3 composite systems [M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1, (1996)]

States which are inseparable, but have positive partial transposition contain so-called "Bound Entanglement" because you cannot distill any pure entangled states from them. (by making measurements on the mixed state, and throwing away those that you don't think will have much entanglement in them)

are there more entangled or mixed states?

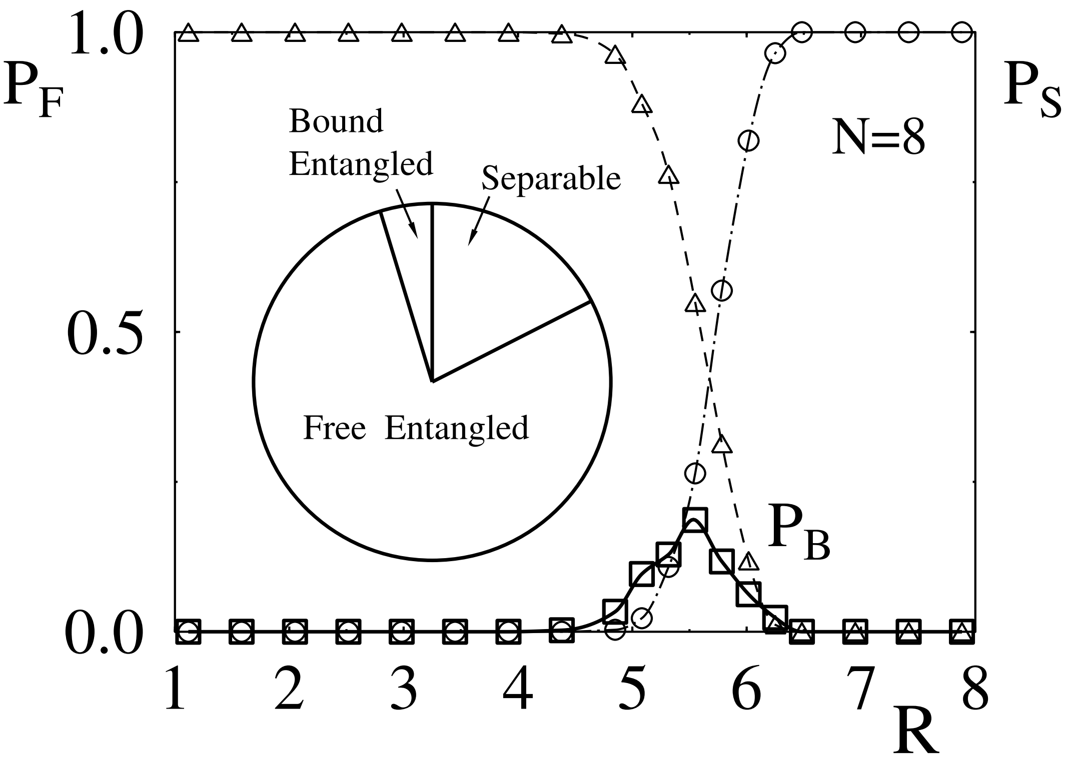
This was recently investigated by [K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Phys. Rev. A **58**, 883 (1998); K. Życzkowski Phys. Rev. A **60**, 3496 (1999)]. Find that as the dimension N of the composite Hilbert space increases, the "volume" of states which have positive partial transposition decreases exponentially.

Numerically they also found that the "volume" of Bound entangled states is small compared to the distillable entangled states for large N.

Figure taken from[K. Życzkowski Phys. Rev. A **60**, 3496 (1999)].

Hilbert Space Dimension $N = 8$

Triangles: Distillable entangled states
Squares: Bound entangled states
Circles: Separable states



$$R = \frac{1}{\mathrm{Tr}\left[\hat{\rho}^2\right]} \qquad P_F = \text{proportion of states}$$

In the paper of Życzkowski *et. al.* it was also shown that all the states in a sufficiently small neighbourhood of the totally random state I/N are separable.

So how big is this neighbourhood?

Some work done at U. Q. and U. of New Mexico by

P. Rungta, W. J. Munro, K. Nemoto, P. Deuar, G. J. Milburn, and C. M. Caves

Looked at density matrices of the form

$$\widehat{\rho} = (1 - \varepsilon)\frac{\widehat{I}}{D^N} + \epsilon\widehat{\rho}_1$$

where $\widehat{\rho}_1$ is some entangled state.

We obtained that for composite systems composed of N subsystems of Hilbert space dimension D, when

$$\varepsilon \leq \frac{1}{1 + D^{2N-1}}$$

all such states are separable

And when

$$\varepsilon > \frac{1}{1 + D^{N-1}}$$

there are some states $\hat{\rho}_1$ that will make the whole state $\hat{\rho}$ entangled.

There was a weaker bound on the latter obtained previously. [G. Vidal and R. Tarrach, Phys. Rev. A **59**, 141 (1999)] when

$$\varepsilon \leq \frac{1}{1 + D^{2N-1}}$$

all such states are separable

ε is related to how easy it is for the addition of random noise to destroy entanglement. The lower the bound, the harder it is to destroy all the entanglement.

Note how ε decreases exponentially with N the number of subsystems but only polynomially with D the Hilbert space dimension of the subsystems.

We've also recently found that an important class of states, the noisy maximally entangled states

$$\widehat{\rho} = (1 - \varepsilon)\frac{\widehat{I}}{D^N} + \epsilon \left|E\right\rangle \left\langle E\right|$$

contain no bound entanglement, and are always distillable.

(Because the exact entanglement-separability point is given by the partial transposition condition)

Thank You