The Wigner Stochastic Gross-Pitaevskii Equation:
a stable c-field theory that includes quantum fluctuations

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Aim: A c-field description that includes quantum fluctuations in the stationary state and evolution

To be used for: (*) Generating a thermal ensemble of single realizations

(**) Calculating the quantum dynamics including nonlinear defects and other single-shot phenomena

Concept

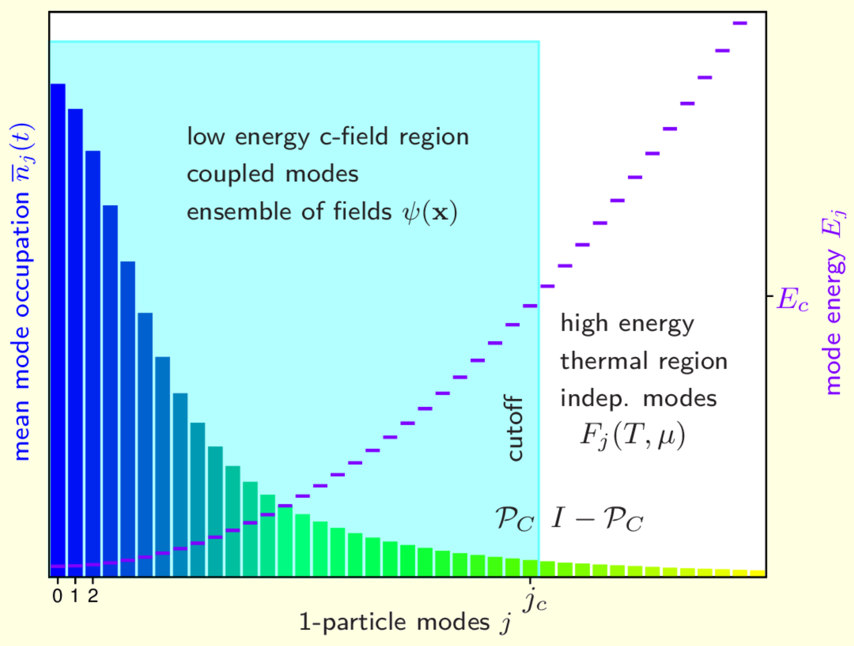
- Re-derive the SGPE (PSGPE) equations from a Wigner representation of the low-energy Bose field
- BUT this time: *without explicitly assuming high occupations*
- Begin like Gardiner+Davis, *J. Phys. B* **36**, 4732 (2003) using the SGPE model.
- High energy tail is a constraint, not a bath
- Still assume linearized Gibbs factors in tail:

$$\exp\left[\frac{\mathcal{H}_{GP}-\mu}{k_BT}\right]\approx 1+\frac{\mathcal{H}_{GP}-\mu}{k_BT}$$

- End up with an ensemble of c-fields:

$$\widehat{\Psi}(\mathbf{x})=\sum_j \widehat{a}_j Y_j(\mathbf{x})\to\sum_{j\leq j_c}\alpha_j Y_j(\mathbf{x})=\psi_W(\mathbf{x})$$

The SGPE model



Projector: $P_C(\mathbf{x},\mathbf{y})=\sum_{j\leq j_c}Y_j(\mathbf{x})Y_j(\mathbf{y})^*\approx\delta(\mathbf{x}-\mathbf{y})$

e.g. plane waves: $Y_j(\mathbf{x})=\frac{1}{\sqrt{V}}e^{i\mathbf{k}_j\cdot\mathbf{x}}$

Existing methods

$$\mathcal{H}_{GP}=-\frac{\hbar^2}{2m}\nabla^2+g|\psi(\mathbf{x})|^2+V(\mathbf{x})$$

The baseline: GPE $\hbar\frac{\partial\psi(\mathbf{x})}{\partial t}=\mathcal{P}_C\Big\{-i\mathcal{H}_{GP}\psi(\mathbf{x})\Big\}$ No quantum fluctuations

Self-thermalizes at long times to a canonical ensemble, T set by cutoff

SGPE (Stochastic GP Equation)

$$\hbar\frac{\partial\psi(\mathbf{x})}{\partial t}=\mathcal{P}_C\Big\{-i\mathcal{H}_{GP}\psi(\mathbf{x})-\gamma\left(\mathcal{H}_{GP}-\mu\right)\psi(\mathbf{x})+\sqrt{2\gamma\hbar k_BT}\,\eta(\mathbf{x},t)\Big\}$$

Stable → GCE at a set T; No quantum fluctuations, assumes macroscopic occupation

Truncated Wigner

$$\hbar\frac{\partial\psi_W(\mathbf{x})}{\partial t}=\mathcal{P}_C\Big\{-i\mathcal{H}_{\rm Wig}\psi_W(\mathbf{x})\Big\}$$

Quantum fluctuations at short time due to initial noise
→ these are later converted to heat by the GPE.
→ Unclear crossover into a stationary state with no quantum fluctuations, and interpretation problems for ψ_W

Positive P $\mathcal{L}_{PP}=-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{x})-\mu+g\,\widetilde{\psi}(\mathbf{x})^*\psi(\mathbf{x})$ Swislocki, Deuar, *J Phys B* **49**, 145303 (2016)

$$\hbar\frac{\partial\psi(\mathbf{x})}{\partial t}=-i(1-i\gamma)\mathcal{L}_{PP}\psi(\mathbf{x})+\sqrt{i\hbar g\,(1-2i\gamma)}\,\xi(\mathbf{x},t)\psi(\mathbf{x})+\sqrt{2\gamma\hbar k_BT}\,\eta(\mathbf{x},t)$$

$$\hbar\frac{\partial\widetilde{\psi}(\mathbf{x})}{\partial t}=-i(1-i\gamma)\mathcal{L}^*_{PP}\widetilde{\psi}(\mathbf{x})+\sqrt{i\hbar g\,(1-2i\gamma)}\,\widetilde{\xi}(\mathbf{x},t)\widetilde{\psi}(\mathbf{x})+\sqrt{2\gamma\hbar k_BT}\,\eta(\mathbf{x},t)$$

Unstable numerically → equilibrium not achieveable; Full quantum mechanics while it lasts

WSGPE evolution equation:

$$\hbar\frac{\partial\psi_W(\mathbf{x})}{\partial t}=\mathcal{P}_C\left\{-i\mathcal{H}_{\rm Wig}\psi_W(\mathbf{x})-\gamma\Big[\mathcal{H}_{\rm Wig}-\mu\Big]\psi_W(\mathbf{x})+\sqrt{\gamma\hbar\Big[2k_BT+\mathcal{H}_{\rm Wig}-\mu\Big]}\,\eta(\mathbf{x},t)\right\}$$

complex thermal noise

Wigner energy functional with explicit dependence on particle number (not just gn): $\mathcal{H}_{\rm Wig}=-\frac{\hbar^2}{2m}\nabla^2+g\Big[|\psi_W(\mathbf{x})|^2-P_C(\mathbf{x},\mathbf{x})\Big]+V(\mathbf{x})$

Extra thermal noise at high energy

Implementation:

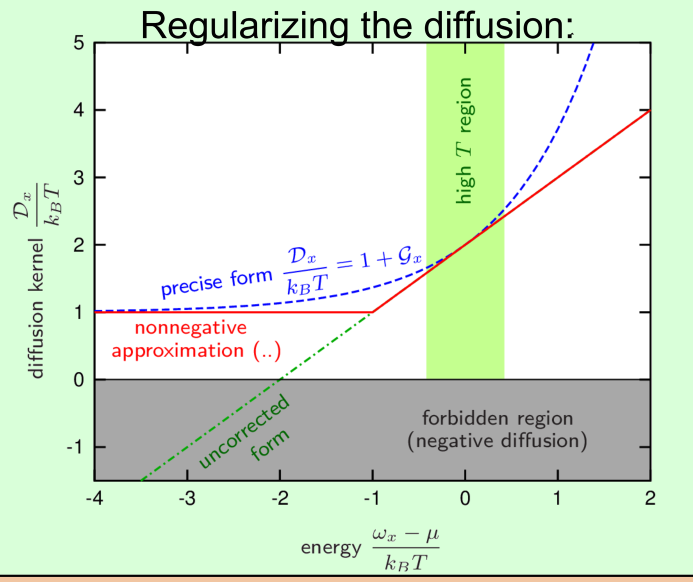
$$\hbar\frac{\partial\psi_W(\mathbf{x})}{\partial t}=\mathcal{P}_C\left\{-i\mathcal{H}_{\rm Wig}\psi_W(\mathbf{x})-\gamma\left[-\frac{\hbar^2}{2m}\nabla^2+\mathcal{D}^{\rm reg}_x(\mathbf{x})-2k_BT\right]\psi_W(\mathbf{x})+\sqrt{\gamma\hbar\,\mathcal{D}^{\rm reg}_x(\mathbf{x})}\,\eta(\mathbf{x},t)+\frac{\sqrt{\gamma\hbar}}{(2\pi)^{d/2}}\int d\mathbf{k}\frac{\hbar|\mathbf{k}|}{\sqrt{2m}}\widetilde{\eta}(\mathbf{k},t)\right.$$

$$\mathcal{D}^{\rm reg}_x(\mathbf{x})=\max\left[2k_BT-\mu+V(\mathbf{x})+g\Big[|\psi_W(\mathbf{z})|^2-P_C(\mathbf{x},\mathbf{x})\Big],\ k_BT\right]$$

x-space noise $\langle\eta(\mathbf{x},t)^*\eta(\mathbf{x}',t')\rangle=\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$

k-space noise $\langle\widetilde{\eta}(\mathbf{k},t)^*\widetilde{\eta}(\mathbf{k}',t')\rangle=\delta(\mathbf{k}-\mathbf{k}')\delta(t-t')$

Regularizing the diffusion:



Observeables: Symmetrically ordered moments (Weyl symbols) $\langle\widehat{\Psi}^\dagger(\mathbf{x})\widehat{\Psi}(\mathbf{y})\rangle=\langle\psi_W(\mathbf{x})^*\psi_W(\mathbf{y})\rangle_{\rm ens}-\frac{1}{2}P_C(\mathbf{y},\mathbf{x})$

$$\langle\widehat{\Psi}^\dagger(\mathbf{x})\widehat{\Psi}^\dagger(\mathbf{y})\widehat{\Psi}(\mathbf{x})\widehat{\Psi}(\mathbf{y})\rangle=\left\langle\left[|\psi_W(\mathbf{x})|^2-\frac{P_C(\mathbf{x},\mathbf{x})}{2}\right]\left[|\psi_W(\mathbf{y})|^2-\frac{P_C(\mathbf{y},\mathbf{y})}{2}\right]-{\rm Re}\Big[\psi_W(\mathbf{x})^*\psi_W(\mathbf{y})\,P_C(\mathbf{x},\mathbf{y})\Big]+\frac{|P_C(\mathbf{x},\mathbf{y})|^2}{4}\right\rangle_{\rm ens}$$

Appearance of antibunching

Scaling:

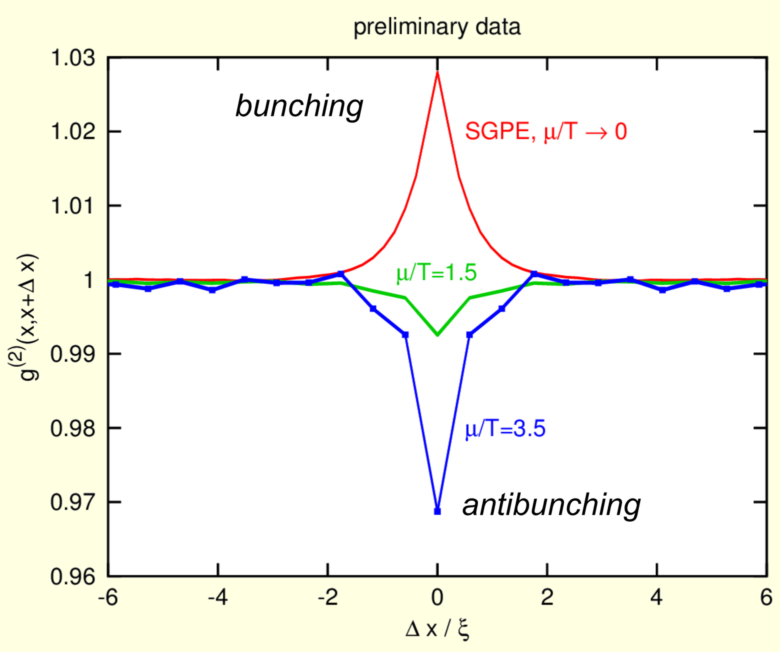
$\mu/T\longrightarrow\lambda\times(\mu/T)$

μ const.

$g\longrightarrow g\lambda$

$\psi_W(\mathbf{x})\longrightarrow\psi_W(\mathbf{x})/\sqrt{\lambda}$

$T\longrightarrow T/\lambda$



TEST CASE: Trapped 1D Bose gas

μ = 22.4, T = 139 = 0.16T_Φ, g=0.01

→ centrally γ=4.5x10^{-6} τ = 5.5x10^{-5}, $\tau/\sqrt{\gamma}$ = 0.026 [cold quasicondensate]

Then, we change g, and $T\sim1/g$, to keep SGPE, μ and $\tau/\sqrt{\gamma}$ constant.

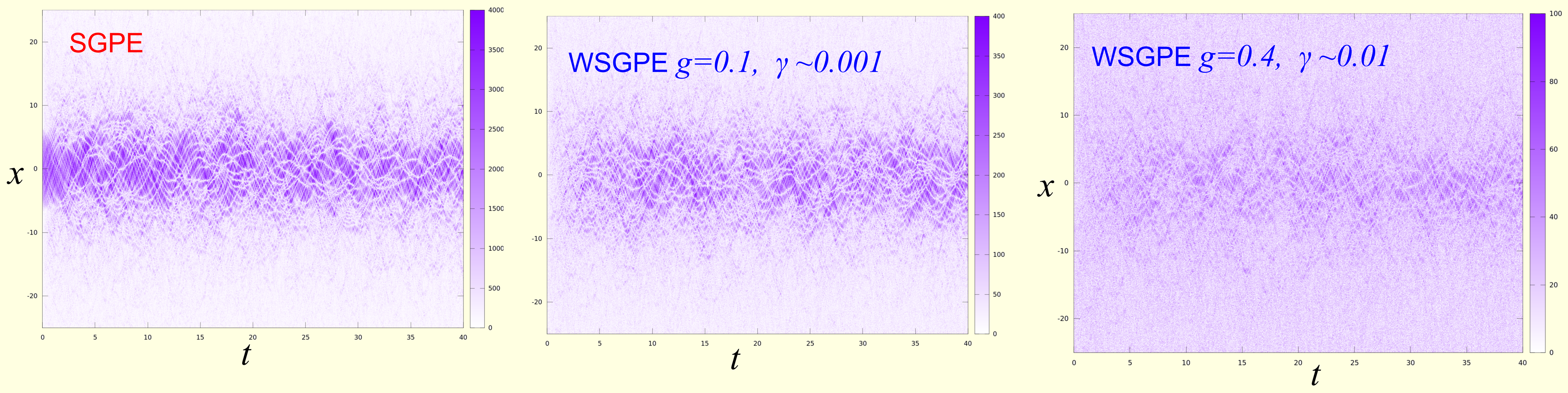
Single shot dynamics

Interaction strength $\gamma=g/n$

Relative temperature $\tau=k_BT/4\pi T_d$

Strong soliton regime $\tau/\sqrt{\gamma}\approx1.0$, μ = 22.4

SGPE has: T = 5000 = 5.6T_Φ g=0.01



Large vacuum fluctuations; but solitons still survive

Local-mode analysis

(Locations in harmonic trap, stationary state)

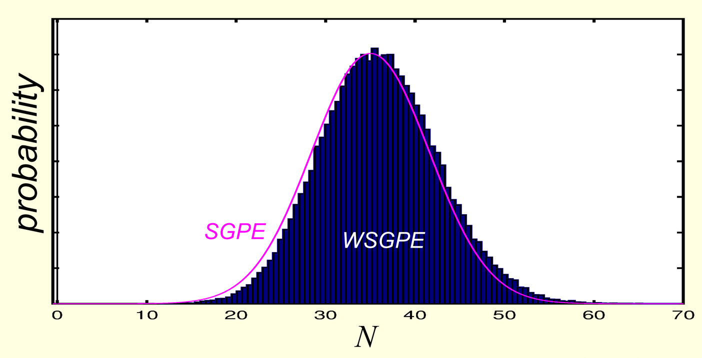
$$\hbar\frac{\partial\psi_W}{\partial t}=-(i+\gamma)\mathcal{L}_{\rm Wig}\psi_W+\sqrt{\gamma\hbar\Big[2k_BT+\mathcal{L}_{\rm wig}\Big]}\,\eta(t)\qquad\mathcal{L}_{\rm Wig}=\frac{m\omega^2x^2}{2}-\mu+g\left(|\psi_W|^2-\frac{1}{\Delta x}\right)$$

High density region $x=0$

$N=\Delta x|\psi_W|^2$

$\Delta E=\frac{m\omega^2x^2}{2}-\mu$

Low density region $x=5a_{ho}$

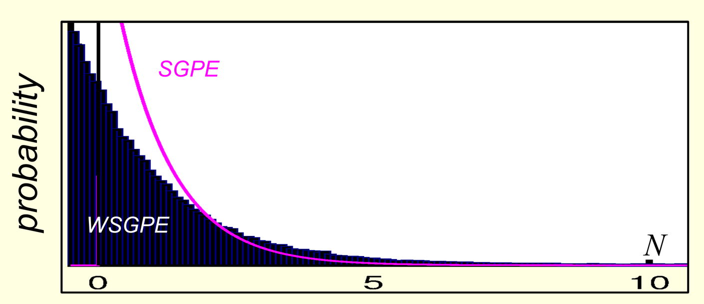


$\langle\hat{a}^\dagger\hat{a}\rangle\approx\frac{N_{TF}-\frac{1}{2}}{g_2}$ $N_{TF}=\frac{\Delta E\Delta x}{g}$ $g_2=\frac{\langle\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}\rangle}{\langle\hat{a}^\dagger\hat{a}\rangle^2}$

Sensible values, density increase from antibunching

main section (small N)

$$P(\psi_W)\approx\exp\left\{-\left(\frac{2\Delta E-\frac{g}{\Delta x}}{2k_BT+\Delta E-\frac{g}{\Delta x}}\right)N-\left(\frac{2\Delta E-\frac{g}{\Delta x}}{(2k_BT+\Delta E-\frac{g}{\Delta x})^2}\right)N^2\right\}$$



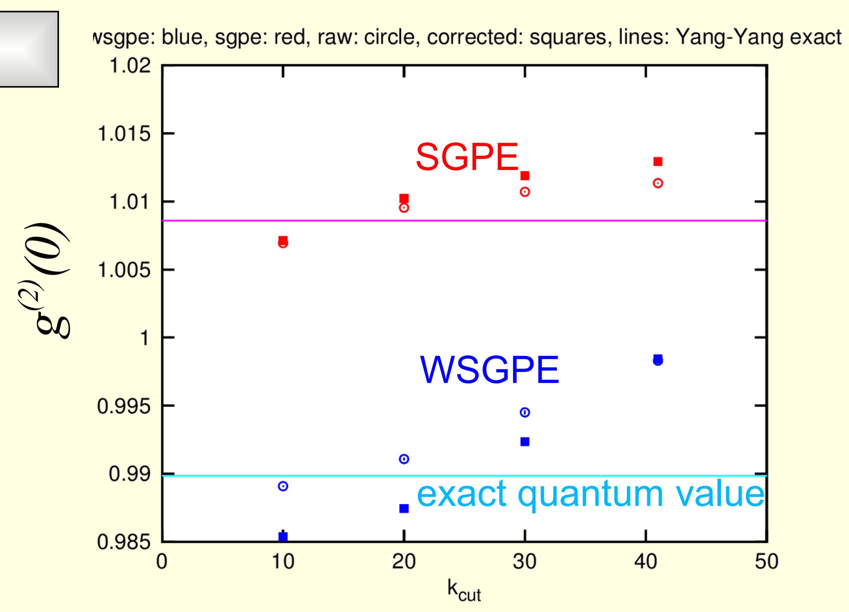
tail section (large N)

$$P(\psi_W)\propto\frac{1}{N}\exp\left[-2N\right]$$

$\langle N\rangle\approx\frac{1}{2}+\frac{k_BT}{\Delta E}$ $\langle\hat{a}^\dagger\hat{a}\rangle\approx\frac{k_BT}{\Delta E}$ *Equipartition (→ cutoff issues)*

Cutoff dependence

Rayleigh-Jeans equipartition remains at high energy due to linearised reservoir coupling



Questions

- What distribution is reached in equilibrium?
- At what point does Wigner truncation affect results?
- Is the cutoff dependence the same as in the SGPE?
- Can the full Gibbs factor be kept without linearization?
- Can the T=0 state contain nonlinear defects?

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