The Wigner Stochastic Gross-Pitaevskii Equation:

a stable c-field theory that includes quantum fluctuations

P. Deuar[1], J. Pietraszewicz[1], N. Proukakis[2], K. Borek[1], T. Świsłocki[1]

[1] Institute of Physics, Polish Academy of Sciences, Warsaw, Poland   [2] Newcastle University, Newcastle on Tyne, United Kingdom

NARODOWE CENTRUM NAUKI · MARIE CURIE ACTIONS · SEVENTH FRAMEWORK PROGRAMME · Ministerstwo Nauki i Szkolnictwa Wyższego · Joint Quantum Centre

Aim: A c-field description that includes quantum fluctuations in the stationary state and evolution

To be used for: (*) Generating a thermal ensemble of single realizations

(**) Calculating the quantum dynamics including nonlinear defects and other single-shot phenomena

Concept

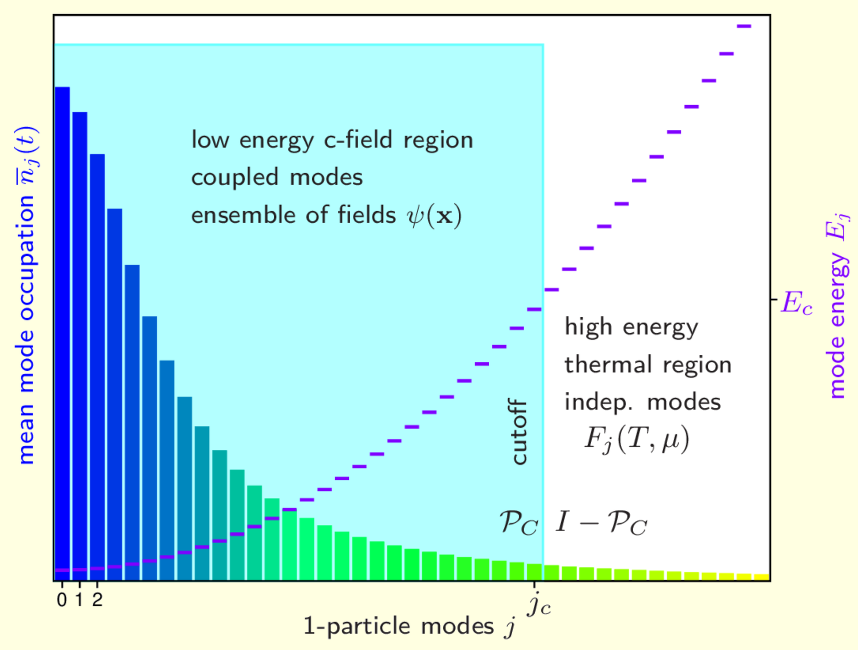
- Re-derive the SGPE (PSGPE) equations from a Wigner representation of the low-energy Bose field
- BUT this time: *without explicitly assuming high occupations*
- Begin like Gardiner+Davis, *J. Phys. B* **36**, 4732 (2003) using the SGPE model.
- High energy tail is a constraint, not a bath
- Still assume linearized Gibbs factors in tail:

$$\exp\left[\frac{\mathcal{H}_{GP}-\mu}{k_BT}\right]\approx 1+\frac{\mathcal{H}_{GP}-\mu}{k_BT}$$

- End up with an ensemble of c-fields:

$$\widehat{\Psi}(\mathbf{x})=\sum_j \hat{a}_j Y_j(\mathbf{x})\to\sum_{j\leq j_c}\alpha_j Y_j(\mathbf{x})=\psi_W(\mathbf{x})$$

The SGPE model



Projector: $P_C(\mathbf{x},\mathbf{y})=\sum_{j\leq j_c}Y_j(\mathbf{x})Y_j(\mathbf{y})^*\approx\delta(\mathbf{x}-\mathbf{y})$

e.g. plane waves: $Y_j(\mathbf{x})=\frac{1}{\sqrt{V}}e^{i\mathbf{k}_j\cdot\mathbf{x}}$

Existing methods

$$\mathcal{H}_{GP}=-\frac{\hbar^2}{2m}\nabla^2+g|\psi(\mathbf{x})|^2+V(\mathbf{x})$$

The baseline: GPE $\hbar\frac{\partial\psi(\mathbf{x})}{\partial t}=\mathcal{P}_C\left\{-i\mathcal{H}_{GP}\psi(\mathbf{x})\right\}$ No quantum fluctuations

Self-thermalizes at long times to a canonical ensemble, T set by cutoff

SGPE (Stochastic GP Equation)

$$\hbar\frac{\partial\psi(\mathbf{x})}{\partial t}=\mathcal{P}_C\left\{-i\mathcal{H}_{GP}\psi(\mathbf{x})-\gamma\left(\mathcal{H}_{GP}-\mu\right)\psi(\mathbf{x})+\sqrt{2\gamma\hbar k_BT}\,\eta(\mathbf{x},t)\right\}$$

Stable → GCE at a set T;   No quantum fluctuations, assumes macroscopic occupation

Truncated Wigner

$$\hbar\frac{\partial\psi_W(\mathbf{x})}{\partial t}=\mathcal{P}_C\left\{-i\mathcal{H}_{\rm Wig}\psi_W(\mathbf{x})\right\}$$

Quantum fluctuations at short time due to initial noise → these are later converted to heat by the GPE. → Unclear crossover into a stationary state with no quantum fluctuations, and interpretation problems for ψ_W

Positive P $\mathcal{L}_{PP}=-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{x})-\mu+g\,\widetilde{\psi}(\mathbf{x})^*\psi(\mathbf{x})$ Swisłocki, Deuar, *J Phys B* **49**, 145303 (2016)

$$\hbar\frac{\partial\psi(\mathbf{x})}{\partial t}=-i(1-i\gamma)\mathcal{L}_{PP}\psi(\mathbf{x})+\sqrt{i\hbar g\,(1-2i\gamma)}\,\xi(\mathbf{x},t)\psi(\mathbf{x})+\sqrt{2\gamma\hbar k_BT}\,\eta(\mathbf{x},t)$$

$$\hbar\frac{\partial\widetilde{\psi}(\mathbf{x})}{\partial t}=-i(1-i\gamma)\mathcal{L}^*_{PP}\widetilde{\psi}(\mathbf{x})+\sqrt{i\hbar g\,(1-2i\gamma)}\,\widetilde{\xi}(\mathbf{x},t)\widetilde{\psi}(\mathbf{x})+\sqrt{2\gamma\hbar k_BT}\,\eta(\mathbf{x},t)$$

Unstable numerically → equilibrium not achieveable;   Full quantum mechanics while it lasts

WSGPE evolution equation:

complex thermal noise

$$\hbar\frac{\partial\psi_W(\mathbf{x})}{\partial t}=\mathcal{P}_C\left\{-i\mathcal{H}_{\rm Wig}\psi_W(\mathbf{x})-\gamma\left[\mathcal{H}_{\rm Wig}-\mu\right]\psi_W(\mathbf{x})+\sqrt{\gamma\hbar\left[2k_BT+\mathcal{H}_{\rm Wig}-\mu\right]}\,\eta(\mathbf{x},t)\right\}$$

Wigner energy functional with explicit dependence on particle number (not just gn): $\mathcal{H}_{\rm Wig}=-\frac{\hbar^2}{2m}\nabla^2+g\left[|\psi_W(\mathbf{x})|^2-P_C(\mathbf{x},\mathbf{x})\right]+V(\mathbf{x})$ Extra thermal noise at high energy

Implementation:

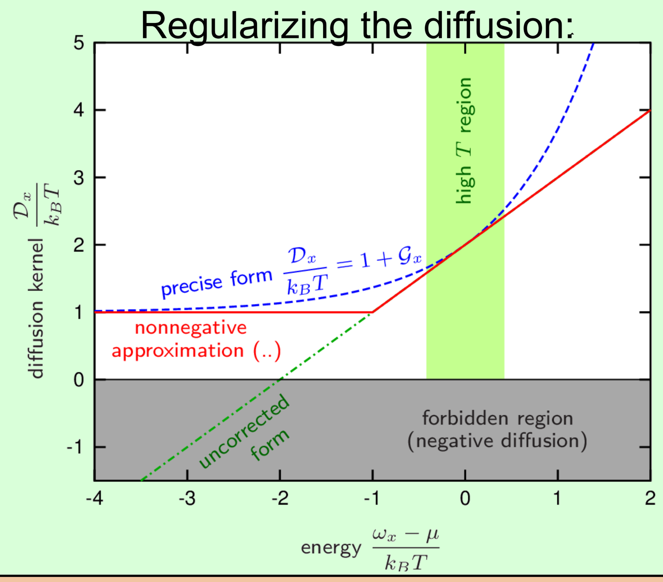
$$\hbar\frac{\partial\psi_W(\mathbf{x})}{\partial t}=\mathcal{P}_C\left\{-i\mathcal{H}_{\rm Wig}\psi_W(\mathbf{x})-\gamma\left[-\frac{\hbar^2}{2m}\nabla^2+\mathcal{D}_x^{\rm reg}(\mathbf{x})-2k_BT\right]\psi_W(\mathbf{x})+\sqrt{\gamma\hbar\,\mathcal{D}_x^{\rm reg}(\mathbf{x})}\,\eta(\mathbf{x},t)+\frac{\sqrt{\gamma\hbar}}{(2\pi)^{d/2}}\int d\mathbf{k}\frac{\hbar|\mathbf{k}|}{\sqrt{2m}}\widetilde{\eta}(\mathbf{k},t)\right.$$

$$\mathcal{D}_x^{\rm reg}(\mathbf{x})=\max\left[2k_BT-\mu+V(\mathbf{x})+g\left[|\psi_W(\mathbf{z})|^2-P_C(\mathbf{x},\mathbf{x})\right],\ k_BT\right]$$

x-space noise $\langle\eta(\mathbf{x},t)^*\eta(\mathbf{x}',t')\rangle=\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$

k-space noise $\langle\widetilde{\eta}(\mathbf{k},t)^*\widetilde{\eta}(\mathbf{k}',t')\rangle=\delta(\mathbf{k}-\mathbf{k}')\delta(t-t')$

Regularizing the diffusion:



Observables: Symmetrically ordered moments (Weyl symbols) $\langle\widehat{\Psi}^\dagger(\mathbf{x})\widehat{\Psi}(\mathbf{y})\rangle=\langle\psi_W(\mathbf{x})^*\psi_W(\mathbf{y})\rangle_{\rm ens}-\frac{1}{2}P_C(\mathbf{y},\mathbf{x})$

$$\langle\widehat{\Psi}^\dagger(\mathbf{x})\widehat{\Psi}^\dagger(\mathbf{y})\widehat{\Psi}(\mathbf{x})\widehat{\Psi}(\mathbf{y})\rangle=\left\langle\left[|\psi_W(\mathbf{x})|^2-\frac{P_C(\mathbf{x},\mathbf{x})}{2}\right]\left[|\psi_W(\mathbf{y})|^2-\frac{P_C(\mathbf{y},\mathbf{y})}{2}\right]-\mathrm{Re}\left[\psi_W(\mathbf{x})^*\psi_W(\mathbf{y})\,P_C(\mathbf{x},\mathbf{y})\right]+\frac{|P_C(\mathbf{x},\mathbf{y})|^2}{4}\right\rangle_{\rm ens}$$

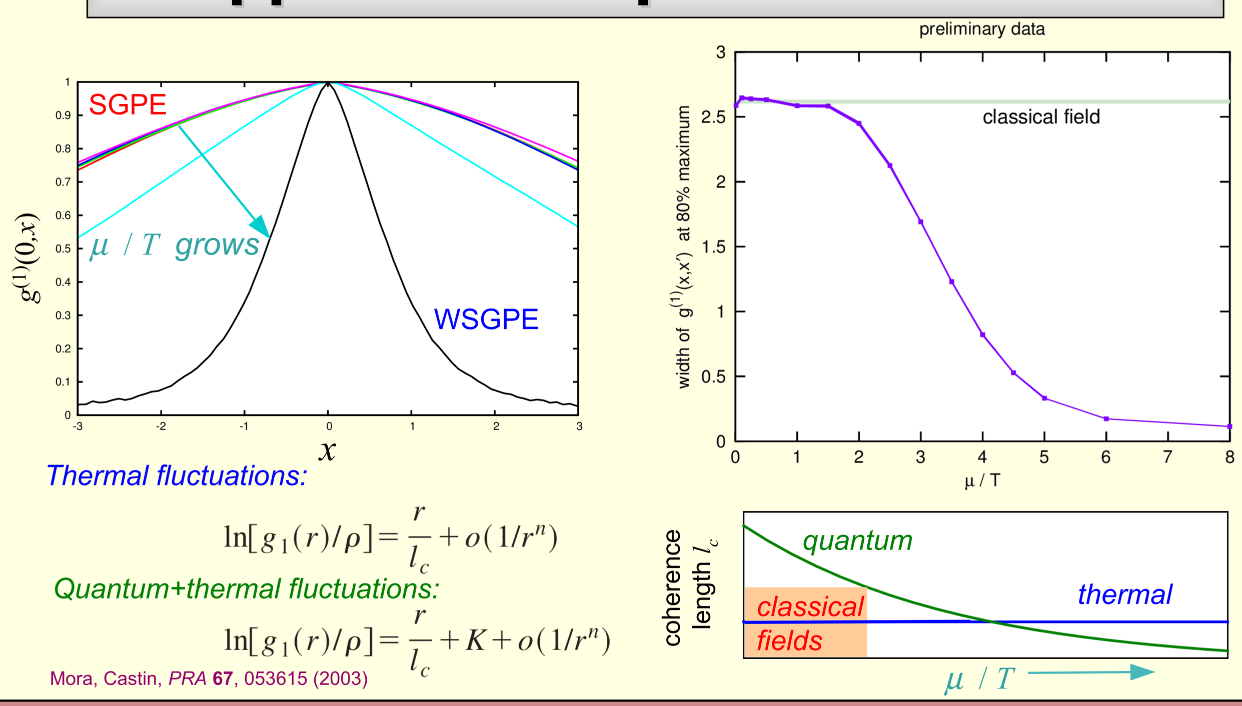
TEST CASE: Trapped 1D Bose gas $\mu=22.4$, $T=139=0.16T_\phi$, $g=0.01$ → centrally $\gamma=4.5\times10^{-6}$ $\tau=5.5\times10^{-5}$, $\tau/\sqrt{\gamma}=0.026$ [cold quasicondensate]

Then, we change g, and $T\sim1/g$, to keep SGPE, μ and $\tau/\sqrt{\gamma}$ constant. Reach $\gamma\sim0.007$ and $\tau\sim0.0022$

Interaction strength $\gamma=g/n$

Relative temperature $\tau=k_BT/4\pi T_d$

Suppression of phase coherence



Scaling:

$\mu/T\longrightarrow\lambda\times(\mu/T)$

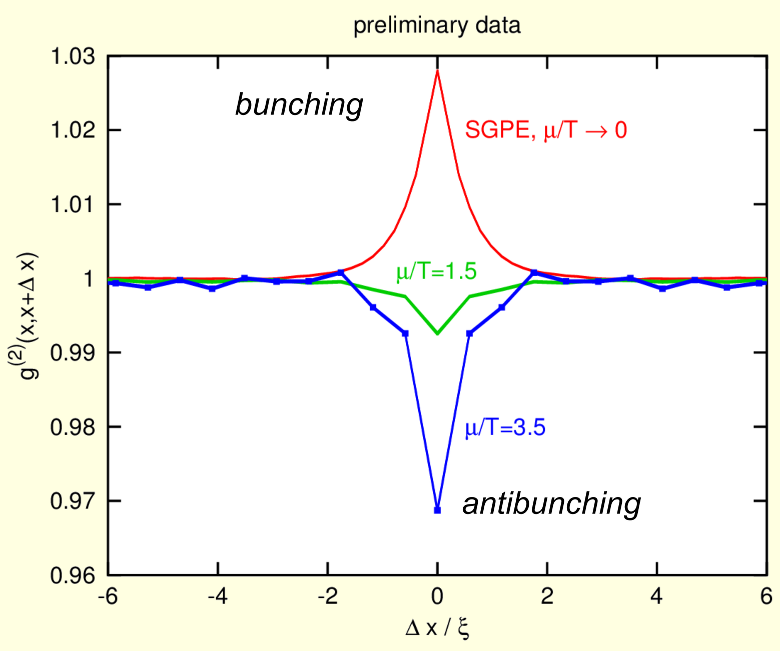
μ const.

$g\longrightarrow g\lambda$

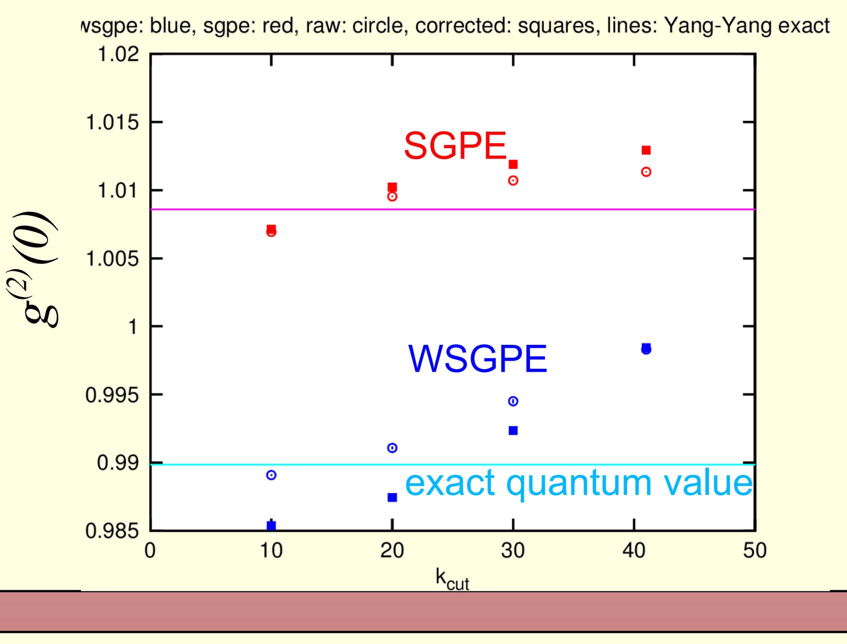
$\psi_W(\mathbf{x})\longrightarrow\psi_W(\mathbf{x})/\sqrt{\lambda}$

$T\longrightarrow T/\lambda$

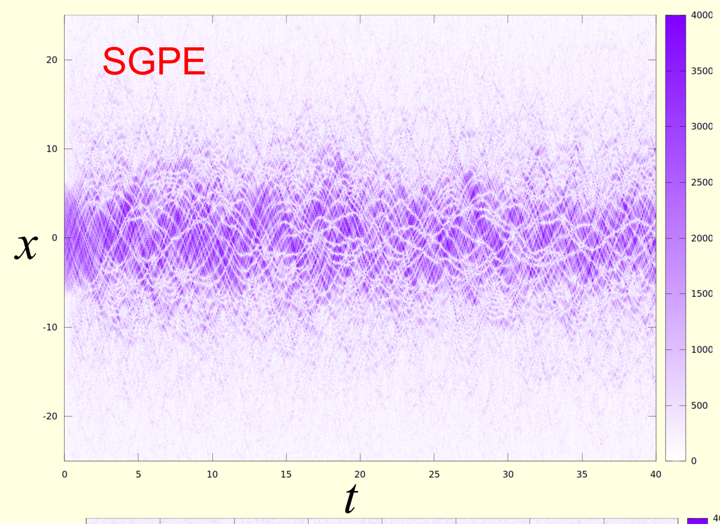
Appearance of antibunching



Cutoff dependence



Single shot dynamics

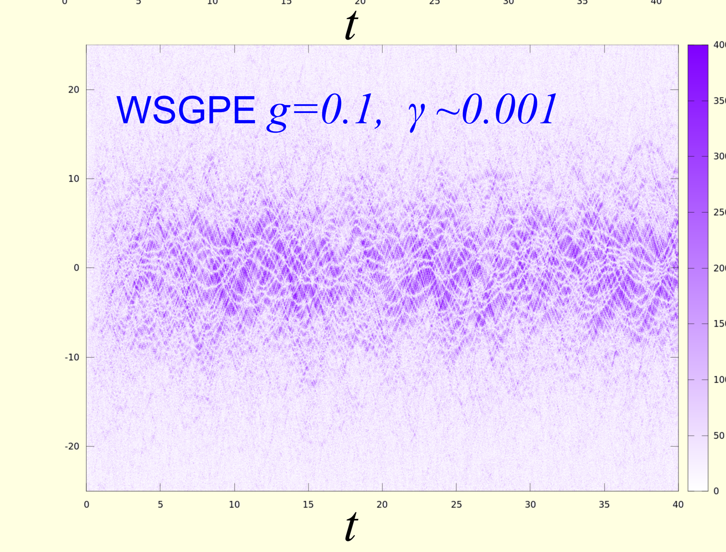


Strong soliton regime

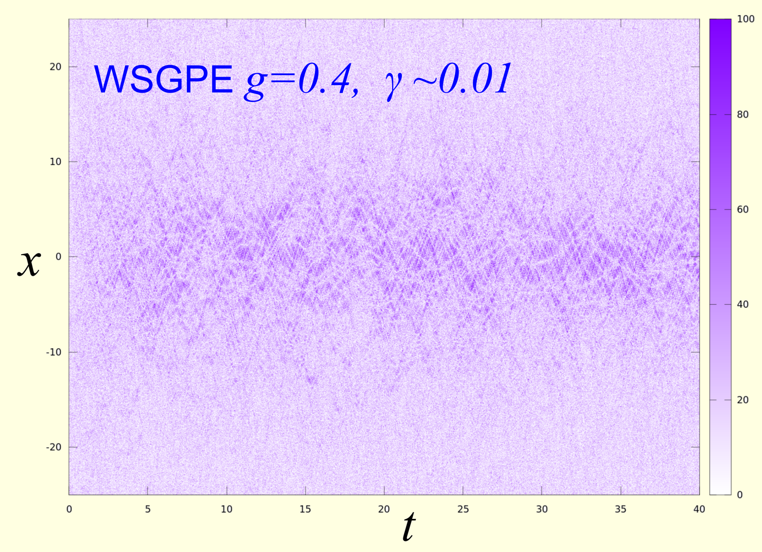
$\tau/\sqrt{\gamma}\approx1.0$, $\mu=22.4$,

SGPE has:

$T=5000=5.6T_\phi$ $g=0.01$



Large vacuum fluctuations;
Solitons still survive

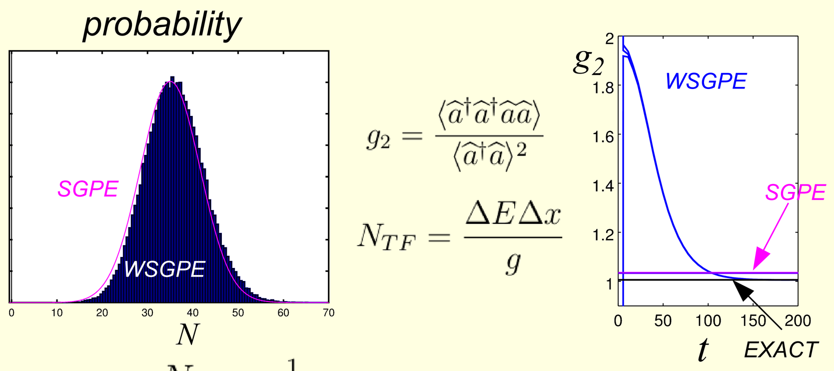


One-mode analysis

(Stationary state)

$$\hbar\frac{\partial\psi_W}{\partial t}=-(i+\gamma)\mathcal{L}_{\rm Wig}\psi_W+\sqrt{\gamma\hbar\left[2k_BT+\mathcal{L}_{\rm wig}\right]}\,\eta(t)\qquad\mathcal{L}_{\rm Wig}=\frac{m\omega^2x^2}{2}-\mu+g\left(|\psi_W|^2-\frac{1}{\Delta x}\right)$$

High density case $x=0$

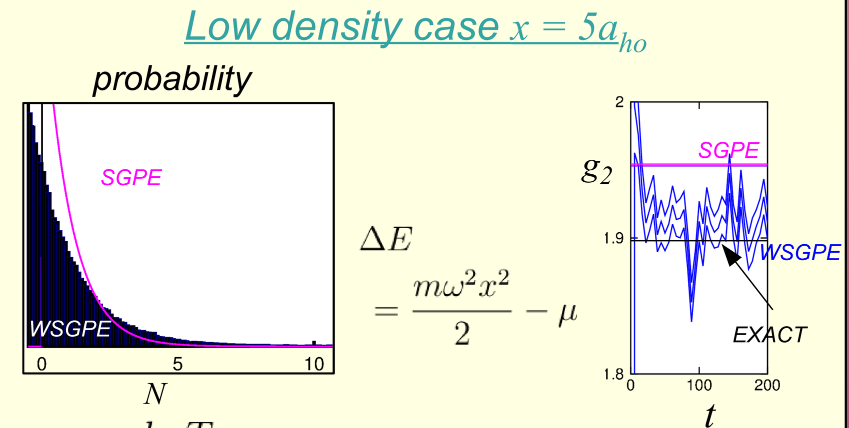


$g_2=\frac{\langle\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}\rangle}{\langle\hat{a}^\dagger\hat{a}\rangle^2}$

$N_{TF}=\frac{\Delta E\Delta x}{g}$

$\langle\hat{a}^\dagger\hat{a}\rangle\approx\frac{N_{TF}-\frac{1}{2}}{g_2}$ Sensible values, density increase from antibunching

Low density case $x=5a_{ho}$



$\Delta E=\frac{m\omega^2x^2}{2}-\mu$

$\langle\hat{a}^\dagger\hat{a}\rangle\approx\frac{k_BT}{\Delta E}-\frac{g}{4\Delta E\Delta x}$ Equipartition (→ cutoff issues) Problems if 2nd term too large

Questions

- What distribution is reached in equilibrium?
- At what point does Wigner truncation affect results?
- Is the cutoff dependence the same as in the SGPE?
- Can the full Gibbs factor be kept without linearization?
- Can the $T=0$ state contain nonlinear defects?

We acknowledge the support of the NCN grant 2012/07/E/ST2/01389