Appendix B

Some details of Stochastic Calculus

Some results regarding random terms in differential equations relevant to the thesis are gathered here. Proofs of these can be found in Gardiner[53], with details on computer algorithms in [82].

In many cases, a set of stochastic differential equations (i.e. differential equations with random terms) can be written in the Langevin form

$$dx_j(t) = A_j(\mathbf{x}, t)\, dt + \sum_k B_{jk}(\mathbf{x}, t)\, dW_k(t), \tag{B.1}$$

where $\mathbf{x}$ contains all variables x_j. The $dW_j(t)$ are Wiener increments and (B.1) is to be interpreted according to the Ito calculus (see below).

A Wiener increment is defined in terms of the Wiener process $W(t)$, which is the special case of (B.1) with $j = 1$, $A_1 = 0$ and $B_{1k} = \delta_{k1}$. The probability distribution of W is governed by the Fokker-Planck equation (FPE)

$$\frac{\partial P_W(W, t)}{\partial t} = \frac{\partial^2 P_W(W, t)}{\partial W^2}, \tag{B.2}$$

a special case of Brownian motion. The individual realizations of W are continuous but not differentiable. The (infinitesimal) Wiener increment can, however, be defined, and is

$$dW(t) = W(t + dt) - W(t), \tag{B.3}$$

with dt infinitesimal. This random quantity has the expectation values

$$\langle dW(t)\rangle_{\text{stoch}} = 0 \tag{B.4a}$$

$$\left\langle dW(t)^2\right\rangle_{\text{stoch}} = dt \tag{B.4b}$$

$$\langle dW(t)dW(t')\rangle_{\text{stoch}} = 0 \qquad \text{if } t \neq t' \tag{B.4c}$$

$$\left\langle \prod_{j=1}^{\max[j]>2} dW(t_j) \right\rangle_{\text{stoch}} = 0 \tag{B.4d}$$

The Wiener increment is related to processes with white noise correlations such that if $\langle \xi(t)\xi(t')\rangle_{\text{stoch}} = \delta(t-t')$ then one can write $\xi(t) = dW(t)/dt$. In (B.1), the Wiener increments are independent: $\langle dW_j dW_{k\neq j}\rangle_{\text{stoch}} = \langle dW_j\rangle_{\text{stoch}} \langle dW_k\rangle_{\text{stoch}}$ etc. In a numerical calculation, the Wiener increment is usually implemented as independent Gaussian random variables ΔW_j at each time step of length Δt with mean zero and variance

$$\langle \Delta W_j \Delta W_k\rangle_{\text{stoch}} = \Delta t \delta_{jk}, \tag{B.5}$$

although other choices of the distribution of ΔW_j are possible provided only that the discrete step analogues of (B.4) are satisfied, as in (B.5).

An equation (B.1) in the Ito calculus is equivalent to the Stratonovich calculus equation

$$\begin{aligned} dx_j(t) &= A_j(\mathbf{x},t)\,dt + \sum_k B_{jk}(\mathbf{x},t)\,dW_k(t) + S_j(\mathbf{x},t) \\ &= dx_j^{\text{Ito}}(t) + S_j(\mathbf{x},t), \end{aligned} \tag{B.6}$$

where the *Stratonovich correction* is

$$S_j(\mathbf{x},t) = -\tfrac{1}{2}\sum_{kl} B_{lk}(\mathbf{x},t)\frac{\partial B_{jk}(\mathbf{x},t)}{\partial x_l}. \tag{B.7}$$

These two forms arise from different ways of defining the integral of the differential equations, both useful. For practical purposes, the main differences are that:

- In the Ito calculus, the time-dependent variables $x_j(t)$ are independent of the same-time Wiener increments $W_k(t)$, while this is not the case in the Stratonovich calculus.

- In the Stratonovich calculus, the x_j obey the usual rules of deterministic calculus, in particular the chain rule. In the Ito calculus one instead has

$$dy(\mathbf{x}) = \sum_j dx_j(\mathbf{x},t)\frac{\partial y(\mathbf{x})}{\partial x_j} + \frac{dt}{2}\sum_{jkl} B_{jk}(\mathbf{x},t)B_{lk}(\mathbf{x},t)\frac{\partial^2 y(\mathbf{x})}{\partial x_j \partial x_l}. \tag{B.8}$$

Because of the first point above, the Ito calculus corresponds to a discrete step integration algorithm with the derivative approximated using values at the beginning of the time interval:

$$x_j(t+\Delta t) = x_j(t) + A_j(\mathbf{x}(t),t)\,\Delta t + \sum_k B_{jk}(\mathbf{x}(t),t)\,\Delta W_k(t). \tag{B.9}$$

Implicit integration algorithms estimate the derivative using quantities evaluated at later times[1]

$$t_\kappa = t + \kappa\Delta t \tag{B.10a}$$

(where $\kappa \in (0,1]$) during the timestep. That is,

$$x_j(t+\Delta t) = x_j(t) + A_j(\mathbf{x}(t_\kappa),t_\kappa)\,\Delta t + \sum_k B_{jk}(\mathbf{x}(t_\kappa),t_\kappa)\,\Delta W_k(t). \tag{B.10b}$$

In this case, one must use a Stratonovich (or Stratonovich-like) form of the drift:

$$dx_j(t) \quad = \quad dx_j^{\text{Ito}}(t) + 2\kappa S_j(\mathbf{x},t) \tag{B.11}$$

to ensure that (B.10b) is the same as (B.9) up to lowest deterministic order Δt. This follows by use of (B.5) on (B.10b). For multiplicative noise it has been found that using a semi-implicit method ($\kappa = \frac{1}{2}$) integration method gives superior numerical stability[82].

[1] A common algorithm for implicitly estimating the $\mathbf{x}(t_\kappa)$ is to iterate (B.10b) several times using the appropriate smaller timestep $\Delta t \to \kappa\Delta t$.