Coherent representation approach to damping of two-level systems

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Abstract

A characteristic function for spin systems is applied to investigate the dissipation and decoherence of one and two two-level atoms. Particular attention is devoted to the Schrödinger cat-like states of two two-level systems. It has been demonstrated that the decoherence time and the phase damping rate depend linearly on the number of degrees of freedom for the maximally entangled initial state. Use of various characteristic functions to describe the decoherence processes is advocated. Two realizations of the SU(2) group are discussed in this context. The first one is based on the homomorphism of SU(2) and the group of rotations, the second one takes as its basis the Schwinger coupled-boson representation of angular-momentum operators as well as the definition of the characteristic function provided by Scully and Wódkiewicz.

1. Introduction

Experimental and theoretical investigations of atomic systems which can be thought of as comprising just two levels have played an important role in quantum physics for several decades. However, recent developments in quantum information theory [1–3] have made these systems more important than ever. Indeed, most of quantum information theory is based on the concept of a qubit which is nothing but a two-level system in the language of quantum informatics. All possible experimental realizations of qubits are based on systems that can be described as being composed of just two levels.

Quantum states, including those of qubits, are usually described by density matrices. In the case of some systems—like free particles or harmonic oscillators—an alternative approach has proved to be illuminative. Namely, the description of quantum states with the help of some
\( c \)-number functions (see, e.g., [4–8]) has provided an insight into the structure of quantum states in general, and the quantum–classical correspondence in particular. These \( c \)-number functions, often called quasi-probability distributions, include Husimi functions, Glauber–Sudarshan \( P \)-functions, Wigner functions, etc. With any of the quasi-probability distribution functions one can associate a characteristic function using a Fourier transformation in the complex plane. In this paper, two characteristic functions will be employed: that which is associated with the Wigner distribution, and that corresponding to the Glauber–Sudarshan distribution. The former can be characterized as related to the symmetric, and the latter as related to the normal ordering of operators [5–7]. Usually, the simple term ‘characteristic function’ (without further specifications) indicates that the characteristic function associated with the Wigner distribution is meant. We adhere to this custom throughout this paper.

All functions mentioned above carry the same information about the quantum state as the density matrix, but in many cases they are easier to interpret than the elements of the density operator. The \( c \)-number functions seem especially convenient in the case of the damped systems considered in the present paper. We particularly emphasize here the usefulness of characteristic functions.

Characteristic functions have been most often used to describe states of the quantum harmonic oscillator. Let us briefly review how they work in that simple case.

Let \( \rho \) be the density matrix of a harmonic oscillator. Let \( x \) and \( p \) denote the position and momentum operators, respectively. The characteristic function \( C(k, r) \) for this system is defined as

\[
C(k, r) = \text{Tr}(\rho \exp(ikx + ipr))
\]

which can also be written as

\[
C(\eta, \eta^*) = \text{Tr}(\rho \exp(i\eta a^\dagger - \eta^* a)),
\]

if we express the coordinate and momentum operators in terms of the creation and annihilation operators and define \( \eta = -\sqrt{m\omega/\hbar}x + i\sqrt{\hbar/(2m\omega)}k \), \( \eta^* = -\sqrt{m\omega/\hbar}x - i\sqrt{\hbar/(2m\omega)}k \).

In the above formula, the symbols \( m \) and \( \omega \) denote the mass and frequency of the harmonic oscillator, respectively. Knowledge of the characteristic function for all values of \( k \) and \( r \) (or \( \eta, \eta^* \)) is equivalent to knowledge of the whole density matrix \( \rho \). The density matrix itself can be expressed in terms of the characteristic functions as an integral [9, 10]:

\[
\rho = \frac{1}{\pi} \int C(\eta, \eta^*) \exp(-\eta a^\dagger + \eta^* a) \, d^2\eta.
\]

The characteristic function allows one to calculate the moments of position and momentum by simple differentiation. However, the advantages of the characteristic function can be more fully understood if one considers the relation between that function and the Weyl group. The Weyl group consists of transformations \( H(k, r) \) generated by position and momentum operators: \( H(k, r) = \exp(ikx + ipr) \). The Weyl group is not the symmetry group, nevertheless it contains fundamental ingredients of quantum physics, that is, the canonical commutation relations between \( x \) and \( p \), and hence the uncertainty relations. The characteristic function is the trace of all Weyl group elements multiplied by the density matrix.

Let us provide here a short list of characteristic functions for simple states of the harmonic oscillator. For the ground state it reads

\[
C(\eta, \eta^*) = e^{-\frac{1}{2} |\eta|^2}.
\]

More generally, for a number state \( |n\rangle \) it is given by

\[
C(\eta, \eta^*) = e^{-\frac{1}{2} |\eta|^2} L_n(|\eta|^2),
\]

where \( L_n \) is the \( n \)th Laguerre polynomial.
For a thermal state characterized by the inverse temperature $\beta_T$ we have

$$C(\eta, \eta^*) = e^{-\frac{1}{2} |\eta|^2} \exp(-|\eta|^2 (e^{\beta_T \hbar \omega} - 1)^{-1}).$$

A coherent state $|\alpha\rangle$ has the following associated characteristic function:

$$C(\eta, \eta^*) = e^{-\frac{1}{2} |\eta|^2} \exp(\eta \alpha^* - \eta^* \alpha).$$

Let us note that the characteristic functions listed above differ just by the exponential factor $\exp\left(-\frac{1}{2} |\eta|^2\right)$ from the characteristic functions associated with the normal ordering of operators as tabulated in [7]. This is a general feature of characteristic functions; see [5, 7].

Finally, for a Schrödinger-cat state $(1/N)(|\alpha\rangle + |-\alpha\rangle)$ (with $N$ being the normalization constant), the characteristic function is equal to

$$C(\eta, \eta^*) = \frac{1}{N^2} e^{-\frac{1}{2} |\alpha|^2} \left[e^{i\eta \alpha^* - i\eta^* \alpha} + e^{-i\eta \alpha^* + i\eta^* \alpha} + e^{-2|\alpha|^2} (e^{i\alpha^* \eta^* \alpha} + e^{-i\alpha^* \eta^* \alpha})\right].$$

We note that it is the exponential factor $e^{-2|\alpha|^2}$ which is responsible for the extremely short decoherence time of the Schrödinger-cat states of electromagnetic modes in cavities [11].

One of the advantages of the characteristic function approach to quantum mechanics is that pure and mixed states are treated on an equal footing: the characteristic function does not discriminate between pure and mixed states. This feature is of particular interest if one considers damping processes where energy is dissipated and the state becomes a mixed state even if at some moment it was a pure state.

It is to be noted that the characteristic function has already been employed for the characterization of decoherence in damped model quantum systems consisting of harmonic oscillators. Of particular relevance is the seminal paper by Unruh and Zurek [12] which was a milestone in the development of the decoherence-based interpretation of quantum mechanics. Another interesting contribution appeared in [13]. However, to the best of our knowledge the characteristic function has not yet appeared as a tool to study dissipation and decoherence in the case of qubits. This is so even though a closely related description in terms of the Wigner distribution for spin systems (as defined in [14]) has been successfully employed in the studies of the interaction of many-atom systems with an electromagnetic field [15], and of the creation of atomic Schrödinger-cat states in cavity QED [16].

The objective of the present work is twofold. Firstly, we show how the characteristic function provides a precise meaning to the idea of the relative phase of two two-level subsystems, a concept which remains somewhat ambiguous without accurate characterization. We also find the characteristic time of phase damping which is the same as the decoherence time of atomic Schrödinger cat-like states. The decoherence time is computed for maximally entangled states of two and $N$ two-level atoms. Secondly, we would like to advocate the usage of characteristic functions as very simple and efficient tools in the analysis of damped spin systems by showing their merits in action.

### 2. Characteristic function approach to a single two-level system

A state of a two-level system is described by a $2 \times 2$ density matrix $\rho$. This matrix has to fulfill all the requirements imposed by quantum mechanics, in particular it has to be Hermitian and of unit trace. Such a matrix is uniquely determined by three real parameters, commonly denoted by $u$, $v$ and $w$. Thus

$$\rho_{11} = \frac{(1 + w)}{2}$$

$$\rho_{22} = \frac{(1 - w)}{2}$$
\[ \rho_{12} = (u - iv)/2 \]
\[ \rho_{21} = (u + iv)/2. \]

The quantity \( w \) is called ‘inversion’, and \( u \) and \( v \) are often referred to as ‘in phase’ and ‘in quadrature’ components of the transition dipole moment, because of their standard interpretation in quantum optics.

Note that the position and momentum operators are not defined for such systems, as opposed to the case of a harmonic oscillator. Therefore, the formalism outlined in the previous section cannot be directly applied here. There is, however, a different group of transformations which for two-level systems plays a role similar to that of the Weyl group for harmonic oscillators. This is the \( SU(2) \) group generated by Pauli matrices \( \vec{\sigma} \). Infinitesimal transformations are generated by the Pauli matrices, and finite \( SU(2) \) transformations are given by application of the operator \( U(\vec{a}) = e^{i \vec{a} \vec{\sigma}} \) where \( \vec{a} = (\sigma_x, \sigma_y, \sigma_z) \). The vector \( \vec{a} \) labels the group element. Because of the well-known relation of the \( SU(2) \) group to the rotation group in three dimensions, \( \vec{a} \) will be called a ‘rotation vector’.

The vector \( \vec{a} \) has the following meaning: the length \( |\vec{a}| \) of \( \vec{a} \) is the angle of rotation while the direction of \( \vec{a} \) determines the axis of rotation. Following the same scheme as in the case of harmonic oscillator we define the characteristic function \( C(\vec{a}) \) as a trace of the \( SU(2) \) operation and the density matrix \( \rho \):

\[ C(\vec{a}) = \text{Tr}(e^{i \vec{a} \vec{\sigma}} \rho). \]

In terms of \( u, v \) and \( w \) the characteristic function can be written as

\[ C(\vec{a}) = \cos(|\vec{a}|) + i \sin(|\vec{a}|) \vec{a} \cdot \vec{U}, \]

where \( \vec{U} \) has components \( U_x = u, U_y = v, U_z = w \). For instance, if the atom is in the ground/excited state, we have \( u = v = 0, w = \mp 1 \), and the characteristic function \( C(\vec{a}) \) is equal to \( \cos(|\vec{a}|) \mp i \sin(|\vec{a}|) \sigma_z \), where the upper sign corresponds to the ground state, and the lower sign to the excited state.

Let us observe here that the naturalness and simplicity of the above expressions can be contrasted with the highly non-trivial expressions for the \( SU(2) \)-Wigner distributions in terms of the density matrix; see [14, 17–20]. Let us follow here the definition of the \( SU(2) \)-Wigner distribution according to Wolf and coworkers [19, 20], which appears to be simpler than that of [14]. We have, by definition,

\[ W^{(W)}(\vec{x}) = \int_{-\pi}^{\pi} h(y) dy \int d\Omega; e^{i\vec{x} \cdot \vec{y}} \text{Tr}(e^{-i\vec{y} \vec{\sigma}} \rho), \]

where \( h(y) \) is a weight function and \( y = |\vec{y}| \). The choice of the weight function is, to some extent, arbitrary. In order to fulfill an important ‘overlap’ condition [19], \( h(y) \) must, however, satisfy certain constraints. In particular, as shown in [19], that condition is satisfied if \( h(y) \) is proportional to the square root of the radial part of the invariant measure on \( SU(2) \), \( h(y) = (1/\sqrt{2})y \sin(y/2) \). Then, with \( x = |\vec{x}| \), the general structure of the Wigner function is quite intuitive:

\[ W^{(W)}(\vec{x}) = 4\pi \left( f_0(x) + f_1(x) \frac{\vec{x} \cdot \vec{U}}{x} \right), \]

but the functions \( f_0(x) \) and \( f_1(x) \) which depend on a particular choice of the weight functions \( h(y) \) as the integrals:

\[ f_0(x) = \int_{-\pi}^{\pi} j_0(xy) \cos(y) h(y) dy, \]
Coherent representation approach to damping of two-level systems

\[ f_1(x) = \int_{-\pi}^{\pi} j_1(xy) \sin(y) h(y) \, dy, \]

are rather complicated:

\[ f_0(x) = 4\sqrt{2} \cos \pi x \frac{4x^2 - 5}{16x^4 - 40x^2 + 9}, \]

\[ f_1(x) = \frac{1}{2\sqrt{2}x} \left[ \frac{16x(4x^2 - 3)}{16x^4 - 40x^2 + 9} \cos \pi x + \text{Si} \left( \pi \left( x - \frac{1}{2} \right) \right) \right. \]

\[ + \text{Si} \left( \pi \left( x + \frac{1}{2} \right) \right) - \text{Si} \left( \pi \left( x - \frac{3}{2} \right) \right) - \text{Si}(\pi(x + 3/2)) \right]. \]

where \( \text{Si}(x) \) is the sine integral [21]. In the preceding formula the symbols \( j_0(xy) \) and \( j_1(xy) \) denote the spherical Bessel functions of the zeroth and first order. Both \( f_0(x) \) and \( f_1(x) \) exhibit damped oscillations for positive \( x \), and both of them become negative in several regions of \( x \).

Thus, it seems to us that it is simpler and more convenient to work directly with the characteristic function rather than with the Wigner function. Obviously, this is not to diminish the value of the ingenious constructions by Agarwal as well as by Wolf and coworkers, which surely deserve further investigation. In section 4, we take advantage of yet another construction of the Wigner function due to Scully and Wódkiewicz [22] who have made use of the elegant Schwinger representation of spin operators in terms of pairs of coupled bosons.

The vector \( \vec{U} \) that appears in equations (8) and (10) can be considered as providing the direction of the spin. There is in fact no better definition of that direction. In the case of a pure state, with \( \text{Tr} \rho^2 = 1 \), the justification is clear: such state is an eigenstate of the operator \( \vec{\sigma} \vec{U} \) with the eigenvalue 1. It is of course very natural to extend the above definition to the case of mixed states.

For a harmonic oscillator, knowledge of the characteristic function for all \( k \) and \( r \) allows for the reconstruction of the density matrix. In the case of a two-level system the situation is not that simple. If \( \rho \) is known then the characteristic function can be calculated with the help of equation (7). For a given characteristic function \( C(\vec{a}) \) one can find the density matrix \( \rho \) using the following relations:

\[ \rho = \int d\mu(\vec{a}) C(\vec{a}) U(\vec{a}). \]

In this equation \( d\mu \) denotes the invariant measure on the \( SU(2) \) group:

\[ d\mu(\vec{a}) = d\Omega_3 \sin^2(|\vec{a}|/2), \]

where \( \Omega_3 \) denotes the directions of the vector \( \vec{a} \).

It should be noted that many characteristic functions may lead to the same density matrix. In fact, there is a one-to-one correspondence between the density matrix and the characteristic function only if the latter has the form given by equation (8). In such a case the characteristic function provides a way of describing the quantum state of the system which is equivalent to the description with the help of density matrix.

Very importantly, the characteristic function can be used to describe damping of a two-level system. This is again because this function does not discriminate between mixed and pure states. The physical origin of damping lies in weak interactions of the system with the reservoir, i.e., a large system with many degrees of freedom. As is well known (see, e.g., [23]), the damping of a two-level system is described by two damping constants: \( \Gamma_1 \) (longitudinal damping constant) and \( \Gamma_2 \) (transverse damping constant). The (interaction-picture) equations that describe the damping are as follows:

\[ w(t) = w(0) \exp(-\Gamma_1 t) + w_\text{eq}(1 - \exp(-\Gamma_1 t)), \]
where \( w_0 \) is the steady-state inversion,

\[
\begin{align*}
    u(t) &= u(0) \exp(-\Gamma_2 t), \\
    v(t) &= v(0) \exp(-\Gamma_2 t).
\end{align*}
\]

(16) \hspace{1cm} (17)

Let us consider the characteristic function of a two-level system damped according to the equations given above. The characteristic function for the initial state is

\[
C(\alpha) = \cos(|\alpha|) + \frac{i}{|\alpha|} \alpha_x w(0) + \frac{i}{|\alpha|} \left( \alpha_x u(0) + i \alpha_y v(0) \right),
\]

(18)

and for \( t > 0 \) it becomes

\[
C(\alpha, t) = C_1(\alpha, t) + C_2(\alpha, t),
\]

(19)

with

\[
\begin{align*}
    C_1(\alpha, t) &= \cos(|\alpha|) + \frac{i}{|\alpha|} \alpha_x w_0 \exp(-\Gamma_1 t) + \frac{i}{|\alpha|} \left( \alpha_x u(0) + i \alpha_y v(0) \right) \exp(-\Gamma_2 t) \\
    C_2(\alpha, t) &= i \frac{\sin(|\alpha|)}{|\alpha|} \left( \alpha_x u(0) + i \alpha_y v(0) \right) \exp(-\Gamma_2 t).
\end{align*}
\]

(20) \hspace{1cm} (21)

It can be seen from the above formula that the characteristic functions tend to the equilibrium form that depends on the steady-state value of the inversion. Initial diagonal components given by \( w(0) \) decay with the characteristic time given by \( \Gamma_1^{-1} \) to the equilibrium value:

\[
C(\alpha, t) = \cos(|\alpha|) + i w_0 \frac{\sin(|\alpha|)}{|\alpha|} \alpha_x
\]

while the initial values of the off-diagonal components \( u \) and \( v \) tend to zero with the time constant provided by \( \Gamma_2 \).

### 3. Two two-level systems

In this section, we will study a system of two two-level atoms. We will concentrate on the entangled states and their damping to the equilibrium separable states.

Let us consider two identical two-level systems. The density matrix \( \rho \) is now a \( 4 \times 4 \) matrix which describes both subsystems including possible correlations between them. The characteristic function of the system will be defined in full analogy to the case of a single two-level atom. Here, however, we need two independent \( SU(2) \) transformations, one for each subsystem. Thus the relevant group is \( SU(2) \otimes SU(2) \). Therefore, in order to define the characteristic function we introduce combined transformations denoted by \( U(\tilde{\alpha}_1, \tilde{\alpha}_2) \). They are tensor products of transformations in each subsystem:

\[
U(\tilde{\alpha}_1, \tilde{\alpha}_2) = \exp(i \tilde{\alpha}_1 \tilde{\sigma}_1) \exp(i \tilde{\alpha}_2 \tilde{\sigma}_2).
\]

(22)

The characteristic function is now defined as

\[
C(\tilde{\alpha}_1, \tilde{\alpha}_2) = \text{Tr}(\rho \exp(i \tilde{\alpha}_1 \tilde{\sigma}_1) \exp(i \tilde{\alpha}_2 \tilde{\sigma}_2)).
\]

(23)

From the quantum information point of view, the most important class of states are entangled states, and our discussion will be restricted to them. In the case of other states that are simply products of individual two-level states, the characteristic function is a product of characteristic functions for each of the states.
Let us now consider the following class of maximally entangled states:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle_1|2\rangle_2 + \exp(i\phi)|2\rangle_1|1\rangle_2).$$

(24)

The first \textit{ket} applies to the first subsystem, the second \textit{ket}—to the second one (as indicated by the subscripts). Based on our intuition we would like to assign the value of \(\phi\) to the relative phase between the two subsystems, but this statement needs some clarification as it is not evident from equation (24) per se.

The characteristic function of the state \(|\psi\rangle\) can easily be found. The result can be expressed as a sum of four terms:

$$C(\vec{a}_1, \vec{a}_2) = \frac{1}{2} \sum_{i=1}^{4} C_i(\vec{a}_1, \vec{a}_2)$$

(25)

where

$$C_1(\vec{a}_1, \vec{a}_2) = \left( \cos(|\vec{a}_1|) - \frac{1}{2} \sin(|\vec{a}_1|) \right) \left( \cos(|\vec{a}_2|) + \frac{1}{2} \sin(|\vec{a}_2|) \right)$$

(26)

$$C_2(\vec{a}_1, \vec{a}_2) = \left( \cos(|\vec{a}_1|) + \frac{1}{2} \sin(|\vec{a}_1|) \right) \left( \cos(|\vec{a}_2|) - \frac{1}{2} \sin(|\vec{a}_2|) \right)$$

(27)

$$C_3(\vec{a}_1, \vec{a}_2) = -\exp(-i\phi) \frac{\sin(|\vec{a}_1|) \sin(|\vec{a}_2|)}{|\vec{a}_1||\vec{a}_2|} (\alpha_{1x} + i\alpha_{1y})(\alpha_{2x} - i\alpha_{2y})$$

(28)

$$C_4(\vec{a}_1, \vec{a}_2) = -\exp(i\phi) \frac{\sin(|\vec{a}_1|) \sin(|\vec{a}_2|)}{|\vec{a}_1||\vec{a}_2|} (\alpha_{1x} - i\alpha_{1y})(\alpha_{2x} + i\alpha_{2y})$$

(29)

We will now concentrate on the characteristic function of phase difference between the two subsystem. In order to do this we will introduce ‘polar coordinates’ for both \(\vec{a}_1\) and \(\vec{a}_2\):

$$\alpha_{1x} + i\alpha_{1y} = |\vec{a}_1| \sin \theta_1 \exp(i\phi_1),$$

(30)

$$\alpha_{1z} = |\vec{a}_1| \cos \theta_1$$

(31)

$$\alpha_{2x} + i\alpha_{2y} = |\vec{a}_2| \sin \theta_2 \exp(i\phi_2)$$

(32)

$$\alpha_{2z} = |\vec{a}_2| \cos \theta_2.$$  

(33)

Let us note that the part of \(C\) that does not depend upon the phase \(\phi\) (given by equations (26) and (27)) does not depend on the phases \(\phi_1\) and \(\phi_2\). The part of \(C\) that depends upon \(\phi\) depends also upon \(\phi_1\) and \(\phi_2\). What is more, if we integrate the characteristic function over all components of \(\vec{a}_1\) and \(\vec{a}_2\) except for that containing the relative phase \(\phi_1 - \phi_2\), we get \(C_{\phi}(\phi_1, \phi_2) = 2 \cos(\phi_1 - \phi_2 - \phi)\). It would be dangerous to say that this formula gives an interpretation of the relative phase \(\phi\) between the subsystems because the characteristic function itself is not a quasi-probability distribution. However, in section 4 we compute a Wigner function which makes such an interpretation valid. What is more, we show the direct parallelism between \(C_{\phi}\) and the corresponding angular dependence of the Wigner function, which suggests that one could venture to interpret \(\phi\) already from knowledge of the characteristic function.

Inspection of equations (25)–(29) shows that the characteristic function contains terms of two kinds. The first term equals \(\cos(|\vec{a}_1|) \cos(|\vec{a}_2|)\) and does not depend upon the directions of vectors \(\vec{a}_1\) and \(\vec{a}_2\). The term depending on the individual components of vectors \(\vec{a}_1\) and \(\vec{a}_2\).
equals:
\[
\frac{\sin(|\vec{\alpha}_1|) \sin(|\vec{\alpha}_2|)}{2|\vec{\alpha}_1| |\vec{\alpha}_2|} (2\alpha_1;\alpha_2; - \exp(-i\varphi)(\alpha_1 x + i\alpha_1 y)(\alpha_2 x - i\alpha_2 y) + c.c.). \tag{34}
\]

We will drop the part depending only on the moduli of $\vec{\alpha}_1$ and $\vec{\alpha}_2$; the rest may be decomposed into three parts: scalar $S$, symmetric traceless $Q$ and antisymmetric $A$,

\[
S = \frac{2 - 4 \cos \varphi}{3}(\alpha_1 \alpha_2 + \alpha_1 \alpha_2 + \alpha_1 \alpha_2)
\]

\[
Q = -\frac{2 + 2 \cos \varphi}{3}(\alpha_1 \alpha_2 + \alpha_1 \alpha_2 - 2\alpha_1 \alpha_2)
\]

\[
A = -2 \sin \varphi(\alpha_1 \alpha_2 - \alpha_1 \alpha_2).
\]

We believe that the above decomposition, and, in particular, symmetric (quadrupole) and the antisymmetric part provide some insight into the directional orientation of the two-spin system in the entangled state.

The interaction of the system with a heat reservoir leads to damping, and the density matrix approaches a steady-state value regardless of the initial state. Damping is a very important process from the point of view of quantum information. All required transformations on qubits should be done during a time shorter than the characteristic decay time. It seems that the most crucial feature of the correlated states, namely the entanglement, is very sensitive to the damping and in some cases may decay with a faster rate than local quantities like individual inversions. This feature makes the process of damping provide a fundamental limit on the possibility of performing quantum operations on qubits.

In this paper, we assume that the two-level subsystems are coupled to individual identical but independent heat reservoirs. However, due to the entanglement, we have to deal with the decay of the relative phase between the subsystems.

As the damping of the individual subsystems is known (it has been described in the previous section), one can easily find the time-dependent characteristic function for the entire system:

\[
C(\vec{\alpha}_1, \vec{\alpha}_2) = \sum_{i=0}^{4} C_i(\vec{\alpha}_1, \vec{\alpha}_2, t)
\]

where

\[
C_1(\alpha_1, \alpha_2, t) = C_1(\alpha_1, \alpha_2) \exp(-\Gamma_1 t)
\]

\[
C_2(\alpha_1, \alpha_2, t) = C_2(\alpha_1, \alpha_2) \exp(-\Gamma_1 t)
\]

\[
C_3(\alpha_1, \alpha_2, t) = C_3(\alpha_1, \alpha_2) \exp(-2\Gamma_2 t)
\]

\[
C_4(\alpha_1, \alpha_2, t) = C_4(\alpha_1, \alpha_2) \exp(-2\Gamma_2 t)
\]

and

\[
C_0(\vec{\alpha}_1, \vec{\alpha}_2, t) = 2(1 - \exp(-\Gamma_1 t)) \left( \cos|\vec{\alpha}_1| - i \frac{\sin(|\vec{\alpha}_1|)}{|\vec{\alpha}_1|} \alpha_1 z \right) \left( \cos|\vec{\alpha}_2| - i \frac{\sin(|\vec{\alpha}_2|)}{|\vec{\alpha}_2|} \alpha_2 z \right).
\]

As before we may integrate this function over all components of $\vec{\alpha}_1$ and $\vec{\alpha}_2$ with exception of those containing the relative phase. In this way we find the time dependence of the mutual phase. The result is

\[
C_{ph}(\varphi_1, \varphi_2) = 2 \cos(\varphi_1 - \varphi_2 - \varphi) \exp(-2\Gamma_2 t).
\]

(40)
Thus the relative phase is damped with a rate larger by a factor of 2 than the transverse damping rate $\Gamma_2$. This shows that the phase relations leading to entanglement are very sensitive to interactions with reservoirs. Even coupling of each subsystem to an individual independent heat bath leads to acceleration of the phase decay.

4. Oscillator-like characteristic function for qubits

The constructions of the characteristic functions as well as the Wigner distribution recalled in the previous sections have one remarkable formal drawback: the standard phase-space structure is completely lost. The characteristic functions do not depend on one or more pairs of variables which could support a canonical Poisson structure. The same is true about the Wigner distributions as defined by either Agarwal or Wolf and coworkers. This situation has been remedied by Scully and Wódkiewicz [22] with the help of Schwinger’s representation of the spin algebra (for an authoritative description of that representation, see, e.g., chapter 3 of [24]).

Let us define the operators

$$J_x = \frac{1}{2}(a + a^\dagger), J_y = \frac{1}{2}(a^\dagger - a), J_z = \frac{1}{2}(a^\dagger - a).$$

The operators $a, a^\dagger, b, b^\dagger$ satisfy canonical commutation relations:

$$[a, a^\dagger] = [b, b^\dagger] = 1, \text{ and all other commutators vanish.}$$

Then the operators $J_x, J_y, J_z$ satisfy the commutation relations of angular momentum. The eigenstates for arbitrary spin $j$ are constructed as

$$|j, m\rangle = (a^\dagger)^j |0\rangle,$$

where $|0\rangle = |0, 0\rangle$ is a rotationally invariant state with zero angular momentum; it plays here the role of a vacuum from which the states with nonzero angular momentum are created with the help of the creation operators $a^\dagger, b^\dagger$. In particular, for $j = 1/2$ the state $|1/2, 1/2\rangle$ can be identified with the excited state of a two-level atom, and the state $|1/2, -1/2\rangle$ with the ground state of that atom. With regard to the commutation relations, the Pauli operators $\sigma_x, \sigma_y, \sigma_z$ can be expressed in terms of $a, a^\dagger, b, b^\dagger$ in the same way as $J_x, J_y$ and $J_z$, except that the factor $1/2$ should be dropped. The general density matrix for the two-level system can be written as

$$\rho = \frac{1}{2}(1 + \vec{\sigma} \cdot \vec{U}),$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Taking this into account, we can define the characteristic functions for a single qubit as

$$\bar{C}(\eta, \eta^*, \xi, \xi^*) = \text{Tr}(\rho \exp((\eta a^\dagger + \xi b^\dagger - \eta^* a - \xi^* b)))$$

(41)

where we have used the ‘bar’ over the sign $C$ of the characteristic function to distinguish it from that used in section 3. Using the above definition, we obtain immediately:

$$\bar{C}(\eta, \eta^*, \xi, \xi^*) = \frac{1}{2} \exp\left[-\frac{1}{2}(|\eta|^2 + |\xi|^2)\right] \times (2 - (|\eta|^2 + |\xi|^2) - (\eta \xi^* + \xi \eta^*)u + i(\eta^* \xi - \eta \xi^*)v + (|\xi|^2 - |\eta|^2)w).$$

(42)

The above formula is clearly analogous to the expression obtained for the characteristic function of a harmonic oscillator in the number state $|n\rangle$, $n = 1$, provided in the introduction (note that $L_1(x) = 1 - x^2$). The general formula for an arbitrary spin appeared in [22].

Using the standard definition of the Wigner function in terms of the characteristic function,

$$\bar{W}(\alpha, \alpha^*, \beta, \beta^*) = \frac{1}{\pi^2} \int \bar{C}(\eta, \eta^*, \xi, \xi^*) \exp[\alpha \eta^* - \alpha^* \eta + \beta \xi^* - \beta^* \xi] d^2\eta d^2\xi,$$

(43)
we obtain immediately
\[
\bar{W}(\alpha, \alpha^*, \beta, \beta^*) = \frac{8}{\pi^2} \exp[-2(|\alpha|^2 + |\beta|^2)] \\
\times \left( |\alpha|^2 + |\beta|^2 - \frac{1}{2} + u(\alpha\beta^* + \alpha^*\beta) + iv(\alpha\beta^* - \alpha^*\beta) + w(|\alpha|^2 - |\beta|^2) \right),
\]
(44)
where, again, the ‘bar’ over the symbol \( W \) of the Wigner function associated with Schwinger’s representation is used to distinguish it from Wolf’s Wigner function \( W^{(W)} \) which appeared earlier.

We can also define a characteristic function \( \bar{C}_{\text{ph}} \) which is obtained from \( \bar{C} \) on integration over the moduli \( |\eta| \) and \( |\zeta| \) and depends only on the difference of the two phases associated with the coupled oscillators.

Indeed, we have
\[
\bar{C}_{\text{ph}}(\psi, \chi) = \int |\eta| d|\eta| \int |\zeta| d|\zeta| \bar{C}(\eta, \eta^*, \zeta, \zeta^*) = -[1 + \pi (u \cos \phi_1 + v \sin \phi_1)],
\]
(45)
where \( \phi_1 = \chi - \psi \), and we have used the following definitions of phases: \( \eta = |\eta| e^{i\phi_1}, \zeta = |\zeta| e^{i\phi_2} \).

Let us now consider the characteristic function associated with the entangled state given by the wavefunction \( |\psi\rangle \) of equation (24). The density matrix corresponding to that wavefunction is equal to
\[
\rho = \frac{1}{2} \left( |1_1\rangle_1 \langle 1_2| \otimes |2_2\rangle_2 \langle 2_1| + e^{-iw}|1_1\rangle_1 \langle 2_2| \otimes |2_2\rangle_2 \langle 1_1| \right) \\
+ e^{iw}|2_1\rangle_1 \langle 1_2| \otimes |2_1\rangle_2 \langle 1_2|,
\]
(46)
where the subscripts standing at the \textit{ket} and \textit{bra} states correspond to the first and second qubit, respectively. As there are two two-level systems, the number of pairs of creation and annihilation operators is equal to 4, and the characteristic function depends on \( \eta_i, \eta_i^* \), and \( \zeta_i, \zeta_i^* \), \( i = 1, 2 \). Direct calculations lead to the following result:
\[
\bar{C}(\eta_i, \eta_i^*, \zeta_i, \zeta_i^*) = \frac{1}{2} \exp \left[ -\frac{1}{2} \sum_{i=1}^{2} (|\eta_i|^2 + |\zeta_i|^2) \right] \\
\times \left[ 2 - \sum_{i=1}^{2} (|\eta_i|^2 + |\zeta_i|^2) + |\eta_1\zeta_1^* + \eta_2\zeta_2^*|^2 + |\eta_1\zeta_1^* - \eta_2\zeta_2^*|^2 e^{2i\phi_1} + |\eta_1\zeta_1^* \zeta_2^* + \eta_2\zeta_2^*|^2 e^{2i\phi_2} + |\eta_1\zeta_1^* \zeta_2^* - \eta_2\zeta_2^*|^2 e^{-2i\phi_1} \right].
\]
(47)

It is again useful to compute the function \( C_{\text{ph}}(\phi_1, \phi_2, \psi) \) which depends on the phases only. After integration of \( \bar{C}(\eta_i, \eta_i^*, \zeta_i, \zeta_i^*) \) over all the moduli, one obtains
\[
C_{\text{ph}}(\phi_1, \phi_2, \psi) = 1 + \frac{\pi}{2} \cos(\phi_1 - \phi_2 - \psi),
\]
(48)
where \( \phi_i = \chi_i - \psi_i \), and the phases \( \psi_i, \chi_i \) are defined from \( \eta_i = |\eta_i| \exp(i\psi_i), \zeta_i = |\zeta_i| \exp(i\chi_i) \). Thus it is clear that the function \( C_{\text{ph}} \) is a useful measure of the phase relations between the components of entangled states. Accordingly, the dynamics of that function may serve as a measure of the time evolution of the related phase.

Dependence of the characteristic function on time is investigated more conveniently if the function based on normal ordering of operators is initially used. Let us define, for \( N \) two-level atoms:
\[
\bar{C}_{\text{norm}}(\eta_i, \eta_i^*, \zeta_i, \zeta_i^*) = \text{Tr} \left[ \rho \prod_{j=1}^{N} e^{i\eta_j a_j} \prod_{j=1}^{N} e^{i\zeta_j b_j} \prod_{j=1}^{N} e^{-\eta_j a_j} \prod_{j=1}^{N} e^{-\zeta_j b_j} \right],
\]
(49)
with \( i = 1, 2, \ldots, N \). The standard (Wigner) characteristic function is obtained from \( C_{\text{norm}} \) by multiplication with the exponential factor:

\[
\exp \left[ -\frac{1}{2} \sum_{i=1}^{N} (|\eta_i|^2 + |\zeta_i|^2) \right].
\]

As before, let us now assume that each two-level system is coupled with its own thermal reservoir characterized by the damping rate \( \gamma_i \), and that the mean number of quanta in the reservoir modes is equal to \( \bar{n}_i \). Then the (interaction-picture) density matrix of the system of \( N \) two-level subsystems satisfies the master equation:

\[
\frac{d\rho}{dt} = \sum_{i=1}^{N} \gamma_i (1 + \bar{n}_i) \left( 2\sigma_+^{(i)} \rho \sigma_-^{(i)} - \sigma_-^{(i)} \sigma_+^{(i)} \rho - \rho \sigma_-^{(i)} \sigma_+^{(i)} \right)
\]

\[
+ \sum_{i=1}^{N} \gamma_i \bar{n}_i \left( 2\sigma_-^{(i)} \rho \sigma_+^{(i)} - \sigma_+^{(i)} \sigma_-^{(i)} \rho - \rho \sigma_+^{(i)} \sigma_-^{(i)} \right),
\]

(50)

where \( \sigma_+^{(i)} = (1/2)(\sigma^{(i)} + i\sigma_0^{(i)}) \), \( \sigma_-^{(i)} = (1/2)(\sigma^{(i)} - i\sigma_0^{(i)}) \).

Using the above formula, we obtain the following simple Fokker–Planck-like equation for the characteristic function \( \tilde{C}_{\text{norm}} \):

\[
\frac{\partial \tilde{C}_{\text{norm}}}{\partial t} = \left[ \sum_{i=1}^{N} \frac{\gamma_i}{2} (1 + \bar{n}_i) \left\{ 2|\xi_i|^2 \frac{\partial^2}{\partial \eta_i \partial \eta_i^*} - \eta_i \frac{\partial}{\partial \eta_i} - \eta_i^* \frac{\partial}{\partial \eta_i^*} \right\} \right. \]

\[
+ \left. \sum_{i=1}^{N} \frac{\gamma_i \bar{n}_i}{2} \left\{ 2|\eta_i|^2 \frac{\partial^2}{\partial \xi_i \partial \xi_i^*} - \xi_i \frac{\partial}{\partial \xi_i} - \xi_i^* \frac{\partial}{\partial \xi_i^*} \right\} \right] \tilde{C}_{\text{norm}}.
\]

(51)

It is interesting to observe that the above equation does not reduce to a first-order partial differential equation even for \( \bar{n}_i = 0 \).

Let us now specialize to the case of the state (24). We look for the solution of equation (51) for \( N = 2 \) in the form:

\[
\tilde{C}_{\text{norm}}(t) = \frac{1}{2} \left[ 2 + \sum_i a_i(t) |\eta_i|^2 + \sum_i b_i(t) |\xi_i|^2 \right.
\]

\[
+ c_1(t) |\eta_1|^2 |\eta_2|^2 + c_2(t) |\eta_1|^2 |\xi_2|^2 + c_3(t) |\eta_2|^2 |\xi_1|^2 \right.
\]

\[
+ c_4(t) |\xi_1|^2 |\xi_2|^2 + d_1(t)|\eta_1|^{\gamma_1}|\xi_2| e^{i\phi} + d_2(t)|\eta_2|^{\gamma_2}|\xi_1| e^{-i\phi} \left].
\]

(52)

Initial conditions for the time-dependent parameters \( a_i, b_i, c_i, d_i \) follow from equation (47):

\[
a_i(0) = b_i(0) = -1, \quad c_1(0) = c_4(0) = 0, \quad c_2(0) = c_3(0) = d_1(0) = d_2(0) = 1.
\]

Straightforward calculations lead to the following solutions:

\[
a_i = -1 + \frac{1}{1 + 2\bar{n}_i} (1 - \exp(-\gamma_i t)), \quad (53)
\]

\[
b_i = -1 - \frac{1}{1 + 2\bar{n}_i} (1 - \exp(-\gamma_i t)), \quad (54)
\]

\[
d_i(t) = \exp \left[ -\frac{1}{2} (\gamma_i + \gamma_2) t \right]. \quad (55)
\]
for \(i = 1, 2,\) and
\[
c_1(t) = \frac{1}{(1 + 2\tilde{\eta}_1)(1 + 2\tilde{\eta}_2)} \left[ 2\tilde{\eta}_1\tilde{\eta}_2 (1 - e^{-\gamma_1 t}) + \tilde{\eta}_1 e^{-\gamma_1 t} (1 - e^{-\gamma_2 t}) + \tilde{\eta}_2 e^{-\gamma_2 t} (1 - e^{-\gamma_1 t}) \right],
\]
\[
c_2(t) = \frac{1}{(1 + 2\tilde{\eta}_1)(1 + 2\tilde{\eta}_2)} \left[ 2\tilde{\eta}_1\tilde{\eta}_2 (1 + e^{-\gamma_1 t}) + \tilde{\eta}_1 e^{-\gamma_1 t} (1 - e^{-\gamma_2 t}) + \tilde{\eta}_2 e^{-\gamma_2 t} (1 + e^{-\gamma_1 t}) \right],
\]
\[
c_3(t) = \frac{1}{(1 + 2\tilde{\eta}_1)(1 + 2\tilde{\eta}_2)} \left[ 2\tilde{\eta}_1\tilde{\eta}_2 (1 - e^{-\gamma_1 t}) + (2 - e^{-\gamma_1 t} - e^{-\gamma_2 t}) \right]
\]
\[
c_4(t) = \frac{1}{(1 + 2\tilde{\eta}_1)(1 + 2\tilde{\eta}_2)} \left[ 2\tilde{\eta}_1\tilde{\eta}_2 (1 + e^{-\gamma_1 t}) + (2 - e^{-\gamma_1 t} - e^{-\gamma_2 t}) \right],
\]
where \(\gamma_i = \gamma_1 (1 + 2\tilde{\eta}_i).\) Thus, the dynamics of populations of the system is governed, in general, by three damping constants: \(\gamma_1, \gamma_2\) and \(\gamma_1 + \gamma_2.\)

The above formulae radically simplify if the two reservoirs are both assumed to have their temperatures equal to zero, and if \(\gamma_1 = \gamma_2.\) In that case we obtain
\[
C_{\text{norm}}(\eta_1, \eta^*, \zeta, \zeta^*) = \frac{1}{2} \{2[1 - e^{-\gamma t})(1 + |\zeta|^2 + |\zeta|^2|\zeta|^2] + (2 - (|\eta_1|^2 + |\eta_l|^2 + |\zeta|^2 + |\zeta|^2) + |\eta_1|^2|\zeta|^2 + |\eta_2|^2|\zeta|^2) e^{-\gamma t} \}
\]
\[
+ (\eta_1 \eta^*_l \zeta \zeta^* e^{i\phi} + \eta^*_1 \eta_2 \zeta \zeta^* e^{-i\phi}) e^{-\gamma t}. \tag{60}
\]

The above expression is in one-to-one correspondence with the formulae (38) and (39) if we note that for the zero-temperature reservoirs we have to deal with the ‘spontaneous emission’ in which case \(\Gamma_1 = 2\Gamma_2 = \gamma.\) Taking into account the last line of equation (60) we again conclude that the rate of decay of the only phase-dependent part of the characteristic function is equal to the sum of decay rates of phases of individual systems.

That last observation of the previous paragraph can trivially be generalized to the case of \(N\) two-level systems. Indeed, let us consider the following entangled state:
\[
|\psi_N\rangle = \frac{1}{\sqrt{2}} \left( |1\rangle_1 |1\rangle_2 \cdots |1\rangle_K |2\rangle_{K+1} |2\rangle_{K+2} \cdots |2\rangle_N \right.
\]
\[
+ e^{i\phi} |2\rangle_1 |2\rangle_2 \cdots |2\rangle_K |1\rangle_{K+1} |1\rangle_{K+2} \cdots |1\rangle_N \right),
\]
with \(0 < K < N.\) The characteristic function corresponding to the above state is easily found to be
\[
\tilde{C}(\eta_l, \eta^*_l, \zeta, \zeta^*) = \exp \left[ -\frac{1}{2} \sum_{j=1}^{K} (|\eta_l|^2 + |\zeta|^2) \right] \tilde{C}_{\text{norm}}(\eta_l, \eta^*_l, \zeta, \zeta^*), \tag{62}
\]
\[
\tilde{C}_{\text{norm}}(\eta_l, \eta^*_l, \zeta, \zeta^*) = \frac{1}{2} \left[ \prod_{j=1}^{K} (1 - |\eta_l|^2) \right] \frac{N}{j=K+1} \left( 1 - |\zeta|^2 \right) + \prod_{j=1}^{K} (1 - |\zeta|^2) \right] \prod_{j=K+1}^{N} (1 - |\eta_l|^2) 
\]
\[
+ (-1)^N \left( e^{i\phi} \sum_{j=1}^{K} \eta^*_l \zeta + e^{-i\phi} \sum_{j=K+1}^{N} \eta^*_l \zeta \right) \right]. \tag{63}
\]
By inspection of formulae (63) and (51) we realize that the dynamics of the phase-dependent parts of the characteristic function completely separate out from the dynamics of the parts which depend on moduli only. Although the latter may be somewhat complicated for non-identical $\gamma_i$ and $\bar{n}_i$, the former exhibit a very simple time dependence characterized by the decay rate:

$$\sum_{i=1}^{N} \frac{1}{2} \gamma_i (1 + 2\bar{n}_i).$$

Thus, the decoherence time of (61) is inversely proportional to the number of degrees of freedom involved. The above observation forms a partial answer to Einstein’s objection against quantum mechanics as expressed in his famous letter to Born of 1954 (the appropriate fragment is quoted, e.g., in [25]). The two components of the wavefunction (61) correspond to ‘narrow’ wavefunctions $\psi_1$ and $\psi_2$ mentioned in that letter. But the superposition (61) is not ‘narrow’, at least in the sense that it has a non-vanishing overlap with a plethora of states located ‘between’ the two components. However, the part of the density operator corresponding to (61) which is responsible for such an overlap decays in time practically immediately when $N$ is large. Similar observations (usually made for systems of harmonic oscillators) have helped to formulate the decoherence-based interpretation of quantum mechanics.

Let us return to the characteristic function for two two-level systems given by equation (60). The corresponding Wigner function, given by

$$\tilde{W}(\alpha_i, \alpha_i^*, \beta_i, \beta_i^*; t) = \frac{1}{\pi^8} \int \tilde{C} \exp \left[ \sum_i (\alpha_i \eta_i^* + \beta_i \zeta_i^* - \alpha_i^* \eta_i - \beta_i^* \zeta_i) \right]$$

(64)
can be calculated without difficulties, and we obtain

$$\tilde{W}(\alpha_i, \alpha_i^*, \beta_i, \beta_i^*; t) = \frac{16}{\pi^4} \exp \left[ -2 \sum_i (|\alpha_i|^2 + |\beta_i|^2) \right]$$

$$\times \left[ (1 - \exp(-\gamma t)) (9 - 12(|\beta_1|^2 + |\beta_2|^2) + 16|\beta_1|^2|\beta_2|^2) \right.$$

$$+ \exp(-\gamma t) (9 - 2(|\alpha_1|^2 + |\alpha_2|^2 + |\beta_1|^2 + |\beta_2|^2))$$

$$+ 8 \exp(-\gamma t) |\alpha_1| |\alpha_2| |\beta_1| |\beta_2| (\exp(i(\mu_1 + v_1 + v_2 - \mu_2 + \varphi)))$$

$$+ \exp(i(-\mu_1 + v_1 - v_2 + \mu_2 - \varphi))) \right],$$

(65)

where the angles $\mu_i, v_i$ are defined via the relations $\alpha_i = |\alpha_i| e^{i\mu_i}$, $\beta_i = |\beta_i| e^{iv_i}$. From the above expression we can find the angular quasi-probability distribution by integrating over $|\alpha_i|, |\beta_i|$: 

$$W_{ph}(\mu_i, v_i; \varphi) = \frac{1}{16\pi^4} \left[ 1 + \frac{\pi^2}{4} e^{-\gamma t} \cos((v_1 - \mu_1) - (v_2 - \mu_2) - \varphi) \right],$$

(66)

which is, naturally, correctly normalized. From the above expression it is clear that the interpretation of $\varphi$ as the phase difference between two qubits is not only possible, but indeed unavoidable. What is more, it could already be formulated from knowledge of the characteristic function, since the expression for $\tilde{C}_{ph}$ has its exact (which means: form-preserving) parallel in $\tilde{W}_{ph}$.

To finish this section, let us observe that, once the Schwinger coupled-boson representation of the angular-momentum operators is introduced, one can also introduce and employ arbitrary $c$-number quasi-probability distributions, including the positive $P$ distribution which seems particularly attractive.
5. Discussion

Deep understanding of entanglement and damping is required to plan qubit manipulations. The decay of entanglement is one of the processes that limit numerous applications of quantum information devices. In some realizations of qubit manipulations the damping plays a marginal role. These include, e.g., trapped ions, where the damping times are long. On the other hand, the entanglement decay is a crucial factor for qubit realizations in the quantum wells in semiconductors. In this case the decay of entanglement takes place at a time scale so short that essentially no qubit manipulation can be performed, even at the lowest available temperatures. It should be noted that the larger the entangled system, the faster the entanglement decay should take place. In this paper, we demonstrated that the coherence between two maximally entangled subsystems is damped with the rate twice as large as the single system rate, and then generalized the results for an entangled system of $N$ qubits.

The decay of entanglement has been the subject of many recent papers. The approach described may be considered as providing an alternative view of the decay of entanglement to that given recently by Yu and Eberly [26]. Those authors have studied the time development of parameters describing the degree of entanglement, and have shown that it may reach zero long before the individual subsystems reach their equilibrium value. Here we show that the decay time constant of the relative phase is larger than the largest time constant of each individual subsystem. Both approaches show that the entanglement is very sensitive to interaction with reservoirs.

Recently, it has been pointed out that the decay of entanglement is never strictly exponential [27], at least because of the quantum interference. In addition, under realistic conditions for qubits in solids, the non-diagonal elements of the density matrix do not decay to zero [28]. Neither of those observations undermine our conclusions on the usefulness of the characteristic functions and the dependence of decoherence time on the number of degrees of freedom.

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