Scattering of narrow stationary beams and short pulses on spheres

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Abstract – A finite-width Gaussian-like beam scattered by a hard and dielectric sphere is shown to be well approximated by a superposition of plane waves, intersecting at small angles and scattered by the sphere. Thus in the present model, the scattering results are described globally by simple analytical formulas. Furthermore, the scattering of short pulses composed of discrete frequency waves on a hard or dielectric sphere is described. This extends the Lorenz-Mie theory for scattering of plane waves.

Introduction. – The description of finite-width beams and their scattering is an important problem for experimental and theoretical studies.

The dedicated, controlled wave beams used in scattering experiments, in comparison to plane-wave application, request lower total energy and produce less noisy signals. It can contribute to more versatile and high-precision measurements useful in scatterers recognition, image reconstructions, computerized tomographic imaging, near-field imaging etc. All these promising and potential applications depend on a reliable description of beams scattering.

An attempt to incorporate beams in scattering theory started long time ago. Starting with the description of a free beam alone [1], the scattering problem has been formulated accordingly. An example of the scattering treatment is presented in [2]. Very inspiring was the discussion given in [3], then continued in [4,5]. These discussions were focused on the definition of beam parameters and basing on that, on finding the scattering results numerically. Thus the discussions attempted to solve the general scattering problem for arbitrary incident beams employing the angular spectrum of plane-waves representation and introducing the corresponding extended boundary conditions. Therefore they were rather complicated and rarely showed distributions of scattered waves [6].

Hard-sphere scalar-beam scattering. – Here we employ the well-known results for a stationary scalar-plane-wave scattering on a hard perfectly reflecting sphere. Superposing such solutions corresponding to different propagation directions, we obtain an exact analytical solution which represents both the incident beam of the prescribed parameters and the scattered beam.

Assuming that the hard and perfectly reflecting sphere of radius \( a \) is placed at the center of the coordinate system and the scalar wave with frequency \( \omega (=k) \) and wavelength \( \lambda = 2\pi/k \) is propagating along the \( z \)-axis, the exact wave function can be written as \( (r > a) \)

\[
\Psi(k, r, \theta, z) = e^{ikz} - \sum_{l=1}^{\infty} i^l (2l + 1) j_l(ka) h_l^1(kr) P_l(\cos \theta),
\]

where \( j_l(\rho) \equiv \sqrt{\frac{\pi}{2\rho}} \, J_{l+\frac{1}{2}}(\rho) \) and \( h_l(\rho) \equiv \sqrt{\frac{\pi}{2\rho}} H_{l+\frac{1}{2}}^{(1)}(\rho) \) are spherical Bessel and Hankel functions, and \( P_l \) are Legendre polynomials. Here and in the following discussion, the units are chosen so that the velocity of light \( c = 1 \).

Our aim is to find such a superposition of the plane waves, propagating along the unit vectors \( \mathbf{e}_s \), see fig. 1, which represents a spatially localized (“Gaussian-like”) beam of its waist \( w \), its centre at \( \mathbf{R}_0 \) and propagating parallel to the \( z \)-axis. The wave function of a superposition of the plane waves is given by \( (r > a) \)

\[
\Psi_0(\mathbf{r}) = \sum_s D_s \exp(-i\mathbf{k}_1 \cdot \mathbf{e}_s) \left\{ \exp(i\mathbf{k} \cdot \mathbf{e}_s) - \sum_{l=1}^{\infty} i^l (2l + 1) \frac{j_l(ka)}{h_l^1(ka)} h_l^1(kr) P_l(\mathbf{e}_s \cdot \mathbf{r}/r) \right\},
\]

where \( D_s \) is the magnitude of the plane wave propagating along \( \mathbf{e}_s \), and the presence of the phase factor

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\[ \text{exp}(-ik\mathbf{R}_0 \cdot \mathbf{e}_s) \] becomes obvious, if the scatterer is taken away \((a \to 0)\).

For the beam in question, the angular spread is given by \(\Delta = \frac{\lambda}{\pi w} \). Within this spread we define \((N + 1) \times (N + 1)\) propagation directions of the plane waves as follows. We first rotate the coordinate system around the \(x\)-axis by the angles \(\delta = \Delta/N\) thus obtaining \(N + 1\) new coordinate systems, and then rotate each of these new systems around the \(y\)-axis, again by \(\delta = \Delta/N\). The unit vectors \(\mathbf{e}_s\) are chosen to coincide with the \(z\)-axis of the rotated coordinate systems. Superposing the plane waves propagating along such \(\mathbf{e}_s\), the resultant wave function will be given by eq. \((1)\) if we replace

\[
\begin{align*}
\mathbf{e}_s \to \mathbf{e}_{ij}, \\
D_s \to D_{ij}, \\
\mathbf{e}_{ij} &= \{-\cos \alpha_i \sin \beta_j, \sin \alpha_i, \cos \alpha_i \cos \beta_j\}, \\
D_{ij} &= \frac{1}{A} \exp \left[ -4 \left( \frac{i - 1}{N} - \frac{1}{2} \right)^2 \right] \exp \left[ -4 \left( \frac{j - 1}{N} - \frac{1}{2} \right)^2 \right],
\end{align*}
\]

where \(A\) is a constant normalization factor, \(\alpha_i = -\frac{\Delta}{2} + (i - 1) \frac{\Delta}{N}, \beta_j = -\frac{\Delta}{2} + (j - 1) \frac{\Delta}{N}, i,j=1,2,...,N+1.\) The factors \(D_{ij}\) are tempering functions, smoothly reducing the more deviated wave components.

**Electromagnetic-beam scattering.** – Even though our discussion concerned the simplest scatterer (hard sphere), this model can easily be generalized to more complex situations, e.g., the Lorenz-Mie scattering by a dielectric sphere, employing the exact scattering field solution given in [7].

The usual descriptions of scattering employ only the part of the wave propagating outward of the scatterer, called the scattered wave [8,9]. The finite-width scattered electromagnetic beams can be obtained, similarly as for the scalar case, by means of the superposition of the total waves created in the vicinity of the scattering object due to the incident plane waves. This total electromagnetic wave in the vicinity of the dielectric spherical scatterer due to the plane circularly polarized electromagnetic wave propagating in the direction of the \(z\)-axis is explicitly given in [7].

Taking an incident, circularly polarized plane wave of frequency \(\omega (= k)\), propagating in the direction of the
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Fig. 4: Scattering pattern in the \((x, z)\)-plane for various displacements of the beam: (a) \(X_0 = 0\). (b) \(X_0 = 5\). (c) \(X_0 = 10\). In all cases, \(a = 5\), \(w = 15\) (see Video2.avi).

Fig. 5: Scattering-diffraction intensity distribution in the \((x, y)\)-planes located at various distances \((Z_0)\) from the scatterer, for the electromagnetic beam \((w = 8)\) scattered on the dielectric sphere \((\epsilon = 2, a = 10\) see Video3.avi).

\[
E_{I_\sigma}^r(k, r) = \frac{1}{\sqrt{2}}(\hat{x} + i \sigma \hat{y}) \exp(ikz),
\]

(2)

the total electric field around a perfect dielectric sphere scatterer is \((r > a)\)

\[
E_{\sigma}(k, r) = E_{I_\sigma}^r(k, r) + \sum_{l=1}^{\infty} \tau_l \left[ c^{TE}_l h_l(kr) X_{l\sigma} 
+ \frac{\sigma}{kr} c^{TM}_l (i \Delta_l Y_{l\sigma} h_l(kr) n 
+ h^D_l(kr) n \times X_{l\sigma}) \right],
\]

(3)

Fig. 6: (Color online) Scattering-diffraction intensity distribution in the \((x, z)\)-plane for scattering with various offsets \((X_0)\) of the electromagnetic beam \((w = 4)\) on the dielectric sphere \((\epsilon = 2, a = 15\) see Video4.avi).

Fig. 7: (Color online) Scattering-diffraction intensity distribution in the \((x, z)\)-plane at the WGM resonance for various offsets of the electromagnetic beam \((w = 1)\) for scattering on the dielectric sphere \((\epsilon = 2, a = 3.937)\). As in fig. 6, the intensity of the resonance strongly depends on the offset (see Video5.avi).
where \( r < a \) we obtain

\[
\mathbf{E}_\sigma(k, r) = \sum_{l=1}^{\infty} \tau_l \left[ a^\text{TE}_l j_l(\sqrt{\kappa}kr) \mathbf{X}_l \sigma + \frac{\sigma}{\kappa r} a^\text{TM}_l (i \lambda_l Y_l j_l(\sqrt{\kappa}kr) \mathbf{n} + j_l^0(\sqrt{\kappa}kr) \mathbf{n} \times \mathbf{X}_l) \right],
\]

(4)

where \( \hat{x} \) and \( \hat{y} \) are unit vectors of the \( x \)- and \( y \)-axis, \( \sigma = \pm 1 \) specifies polarization of the wave, \( \mathbf{n} = r/\rho \), \( \tau_l = i^l \sqrt{2\pi(l+1)} \), \( \lambda_l = \sqrt{l(l+1)} \), \( Y_l \equiv Y_l(\theta, \phi) \) and \( \mathbf{X}_l \equiv \mathbf{X}_l(\theta, \phi) \) are spherical and vector spherical harmonic functions normalized as in [10], and \( j_l^0(\rho) \) and \( h_l^0(\rho) \) are derivatives of the Riccati-Bessel and Riccati-Hankel functions.

The expansion coefficients of transverse electric (TE) and transverse magnetic (TM) components are given by

\[
a^\text{TE}_l(ka) = \frac{j_l(ka)h^0_l(ka) - j_l^0(ka)h_l(ka)}{h^0_l(ka)j_l(\sqrt{\kappa}ka) - h_l(ka)j_l^0(\sqrt{\kappa}ka)},
\]

(5)

\[
a^\text{TM}_l(ka) = \frac{j_l(ka)h^0_l(ka) - j_l^0(ka)h_l(ka)}{\epsilon h^0_l(ka)j_l(\sqrt{\kappa}ka) - \epsilon h_l(ka)j_l^0(\sqrt{\kappa}ka)},
\]

(6)

and the external coefficients are

\[
c^\text{TE}_l(ka) = \frac{j_l(ka)j_l^0(\sqrt{\kappa}ka) - j_l^0(ka)j_l(\sqrt{\kappa}ka)}{h^0_l(ka)j_l(\sqrt{\kappa}ka) - h_l(ka)j_l^0(\sqrt{\kappa}ka)},
\]

(7)

\[
c^\text{TM}_l(ka) = \frac{j_l(ka)j_l^0(\sqrt{\kappa}ka) - j_l^0(ka)j_l(\sqrt{\kappa}ka)}{\epsilon h^0_l(ka)j_l(\sqrt{\kappa}ka) - \epsilon h_l(ka)j_l^0(\sqrt{\kappa}ka)}.
\]

(8)

For the superposition of the plane waves we now obtain

\[
\mathbf{E}_{\text{EB}}(k, r) = \sum_s D_s \exp(-ik\mathbf{R}_0 \cdot (\hat{T}_s \cdot \hat{z})) \hat{T}_s^{-1} \cdot \mathbf{E}_\sigma(\hat{T}_s \cdot \mathbf{r}),
\]

(9)

where \( \hat{z} \) is the unit vector of the \( z \)-axis, and one should replace \( \hat{T}_s \rightarrow \hat{T}_{ij} \),

\[
\hat{T}_{ij} = \begin{bmatrix}
\cos \beta_j & \sin \alpha_i \sin \beta_j & -\cos \alpha_i \sin \beta_j \\
0 & \cos \alpha_i & \sin \alpha_i \\
\sin \beta_j & -\sin \alpha_i \cos \beta_j & \cos \alpha_i \cos \beta_j 
\end{bmatrix},
\]

(10)

leading to

\[
\hat{T}_{ij}^{-1} = \begin{bmatrix}
\cos \beta_j & 0 & \sin \beta_j \\
\sin \alpha_i \sin \beta_j & \cos \alpha_i & -\sin \alpha_i \cos \beta_j \\
-\cos \alpha_i \sin \beta_j & \sin \alpha_i & \cos \alpha_i \cos \beta_j 
\end{bmatrix}.
\]

(11)

The results for scattering of the steady-state scalar beam on the hard and perfectly reflecting sphere are shown figs. 2–4. Similar results but for the electromagnetic beams scattered on the dielectric sphere are given in figs. 5–7. In the situations shown in figs. 5 and 6, the sphere is a kind of lens and focusing of the beam can be seen. In fig. 7, the results for the Whistling Gallery Mode resonance scattering are presented. Large electromagnetic fields are produced inside the sphere, in close vicinity of the surface. This effect strongly depends on the beam offset \( X_0 \).
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Fig. 10: (Color online) Time evolution of a narrow short pulse \((w = 5, t_P = 20, \text{ the offset } X_0 = 8)\) scattered by the dielectric sphere \((\epsilon = 2, a = 15)\), see [Video8.avi].

Fig. 11: (Color online) Time evolution of a wide and very short pulse \((w = 2000, t_P = 3.3)\) scattered by the dielectric sphere \((\epsilon = 2, a = 15)\). The reflected and refracted pulses with single and double internal reflection can be seen (see [Video9.avi]).

Fig. 12: (Color online) Time evolution of very long and very narrow electromagnetic pulse \((t_P = 10^4, w = 1, \text{ the offset } X_0 = 4)\) scattered by the dielectric sphere. The radius \(a (=3.937)\) is chosen so as to excite the WGM resonance \(\text{TE}_{30}\). The excitation and the following decay of the resonance can be seen (see [Video10.avi]).

Scattering of electromagnetic and scalar wave pulses. – The short pulses in our discussion will result from superposition of a finite number of scattered beams with frequencies \(\omega_j\) belonging to some frequency band \(\Delta \omega\). The pulse duration time is \(t_P = 1/\Delta \omega\). The units of length and time are \(\lambda_0 = 2\pi/k_0\) and \(T_0 = 2\pi/\omega_0\), where \(\omega_0 (= k_0)\) is the central (carrier) frequency of the pulse. The basic relations now are \((j = 0, 1, \ldots, N_p)\):

\[
\omega_j = k_j - \omega_0 \left[ 1 + \Delta \omega \left( -\frac{1}{2} + \frac{j}{N_p} \right) \right], \quad (12)
\]

\[
\Psi_P(\Delta \omega, \mathbf{r}, t) = \frac{1}{\sqrt{N_p}} \sum_{j=0}^{N_p} e^{-i\omega_j(t-t_0)} \Psi_B(k_j, \mathbf{r}), \quad (13)
\]

\[
E_{\sigma P}(\Delta \omega, \mathbf{r}, t) = \frac{1}{\sqrt{N_p}} \sum_{j=0}^{N_p} e^{-i\omega_j(t-t_0)} E_{\sigma B}(k_j, \mathbf{r}). \quad (14)
\]

The results for scattering of the pulses are shown in figs. 8–12. In fig. 8, the scalar pulse is scattered on the hard sphere and the remaining figures refer to scattering of the electromagnetic pulse on the dielectric sphere.

Final remarks. – Verifying the above-given solutions and relevant numerical programs, I checked that the required boundary condition (vanishing of the wave function or continuity of tangent components of the fields) was satisfied with the relative accuracy \(10^{-12}\), for calculations done in double precision. The computations were very fast and each single frame of the presented examples took only a few minutes of the PC computing time.

Note that superposing only a finite number of plane waves \([N + 1] \times (N + 1)\) to represent the beam or a finite
number of frequency samples $N_p$, one gets the solutions which are periodic in space and time. In order to remove this recurrence beyond the spatial and temporal scale of experiments, one must take $N$ and $N_p$ sufficiently large ($N, N_p > 10$).

We hope that the presented methods of beam scattering can be applied for example in computer tomography, where by employing numerous scattering data, images of invisible objects can be retrieved with high precision. The present computer tomography methods are mainly based on geometrical optics and straight ray propagation thereby ignoring any diffraction effects. There is a hope [11] that taking into account the wave properties of participating radiations, their diffractions can lead to further improvement of computer tomography, particularly in recognition of small objects of a size comparable to the wavelength.

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