Orbital physics of polar Fermi molecules

Omijoti Dutta, 1 Tomasz Sowiński, 1,2 and Maciej Lewenstein 1,3

1ICFO—Institut de Ciencies Fotòniques, Av. Carl Friedrich Gauss, num. 3, 08860 Castelldefels (Barcelona), Spain
2Institute of Physics of the Polish Academy of Sciences, Al. Lotników 32/46, 02-668 Warsaw, Poland
3ICREA—Institució Catalana de Recerca i Estudis Avançats, Lluis Companys 23, E-08010 Barcelona, Spain

We study a system of polar fermions in a two-dimensional optical lattice and show that the multiband Fermi-Hubbard model is necessary to discuss its properties. We take into account both onsite and long-range interactions between different bands, as well as occupation-dependent inter- and intraband tunnelings. For strong-enough dipolar interactions we predict the appearances of phases such as multiband crystals, smectic metal, and exotic p-wave supersolids.

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I. INTRODUCTION

Creation of ultracold heteronuclear molecules opens the path towards experimental realization of strongly interacting dipolar many-body systems. Depending on the constituent atoms, in moderate electric field these molecules can have a large dipole moment of 1 D in their vibrational ground states [1–4]. In particular, fermionic molecules in the presence of an optical lattice can be used to simulate various quantum phases, such as quantum magnetism and phases of t-J-like models [5–7], various charge density wave orders [8,9], bond-order solids [10], and so on. One should also stress that in the strongly correlated regime, in both bosonic and fermionic systems, the standard descriptions of single-band Hubbard model ceases to be valid. The effect of nonstandard terms become important, leading to novel phases like pair superfluidity and so on [11–16].

While most of the works have been dealing with higher bands concentrated on bosonic systems, in this paper, we study dipolar fermions confined in a 2D optical lattice $V_{\text{lat}} = V_0[\sin^2(\pi x/a) + \sin^2(\pi y/a)] + n\Omega^2 z^2$, where $V_0$ is the lattice depth, $a$ is the lattice constant, $m$ is the mass of the molecule, and $\Omega$ is the frequency of harmonic potential in the $z$ direction. $n$ is the fermion filling in the lattice. The dipoles are polarized along the direction of harmonic trapping. Usually, at low temperature and for low tunneling, the phase diagram consists of different crystal structures whose dependence on the filling $n$ [8]. In this paper, we derive a Fermi-Hubbard model for dipolar fermions, including the effects of higher bands. We show that, even for moderate dipolar strength, it is necessary to take into account the excitations along the $z$ direction. Simultaneously, in this regime, the interaction-induced hopping along the lattice is also important. This changes the phases expected for a so-called spinless Fermi-Hubbard model including only a single band. We want to point out that, as explained later, our model is not equivalent to the models with two distinguishable fermionic species.

Near $n \gtrsim 1/4$, we find a spontaneous appearance of non-Fermi liquid behavior in the form of smectic metallic phase. Near $n \gtrsim 1/2$, we find that the system can be mapped to an extended pseudo-spin-1/2 Hubbard model with a different emergent lattice configuration. We find a regime where chiral $p$-wave superconductivity emerges through a Kohn-Luttinger (KL) mechanism with a transition temperature $T_c$ of the order of tunneling, the temperature where tunneling becomes likely. This gives rise to an exotic supersolid, with the diagonal long-range order provided by the checkerboard pattern of the lower orbital fermions, while the superfluidity originating from the fermions in the higher band.

The paper is organized as follows: In Sec. II we have introduced a multiorbital model to describe dipolar fermions in optical lattices. We then discuss quantitatively the contributions of different parameters present in the model. In Sec. III we have described the energy contribution of different crystal structures in the limit of vanishing tunneling. We also compare the corresponding energies of such crystal states without taking into account the higher bands and show that it is necessary to take into account the higher band contributions for experimentally realizable parameters. In Sec. IV, we have investigated the ground-state properties for filling greater than 1/4. We find that due to the higher band-occupation-dependent tunneling contributions, within a certain parameter regime, there is a spontaneous formation of the smectic-metal phase, along with stripelike phases. In Sec. V we describe the ground-state structures for $n \gtrsim 1/2$. We find that the higher-band tunneling can give rise to sublattices which further can give rise to $p$-wave superfluidity. In Sec. VI we present our conclusions followed by acknowledgments in Sec. VII.

II. MODEL

The Hamiltonian for the dipolar fermions in the second quantized form reads

$$H = \int d^3r [\Psi^\dagger(r)H_0\Psi(r) + \frac{1}{2}\int d^3r d^3r'\Psi^\dagger(r)\Psi^\dagger(r')\mathcal{V}(r-r')\Psi(r')\Psi(r)],$$

where $\Psi(r)$ is a spinless fermion field operator satisfying anticommutation relations $[\Psi(r), \Psi^\dagger(r')] = \delta(r-r')$ and $[\Psi(r), \Psi(r')] = 0$. In the units of recoil energy $E_R = \pi^2\hbar^2/(2ma^2)$, the lattice confinement in the $z$ direction is characterized by the dimensionless parameter $\kappa = \hbar\Omega/(2E_R)$, and the single-particle Hamiltonian becomes $H_0 = -\nabla^2 + V_{\text{lat}}(r)/E_R$. The long-range interaction potential reads $\mathcal{V}(r) = D(1/r^3 - 3z^2/r^5)$, where $D = 2\pi md^2/(\hbar^2a)$ is a dimensionless dipolar strength, related to the electric dipolar moment $D$. 

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d. For KRb molecules with a dipole moment of 0.5 D confined in the optical lattice with \( a = 345 \text{ nm} \) [17] one gets \( D = 8.6 \), whereas, for similar lattice parameters, LiCs molecules can have a dipole moment of \( \sim 5 \) D with \( D \sim 100 \).

We decompose the field operator in the basis of Wannier functions in the \( x \) and \( y \) directions and of harmonic oscillator eigenstates in the \( z \) direction. For convenience, we introduce the orbital index \( \sigma = \{ pml \} \) denoting \( p, m, \) and \( l \) excitations in the \( x, y, \) and \( z \) directions, respectively. In this basis the field operator \( \Psi(r) = \sum_{i,\sigma} \hat{a}_{\sigma i} \Psi_{\sigma i}(r) \), where \( \Psi_{\sigma i}(r) \) is the single-particle wave function in orbital \( \sigma \) localized on site \( \hat{i} = i_x e_x + i_y e_y \) (\( e_x \) and \( e_y \) are unit vectors in the proper directions). Fermionic operator \( \hat{a}_{\sigma i} \) annihilates particle in this state. They satisfy fermionic anticommutation relations, \( \{ \hat{a}_{\sigma i}, \hat{a}_{\sigma j}^\dagger \} = \delta_{\sigma \sigma'} \delta_{ij} \). The Hamiltonian can be rewritten in the following Hubbard-like form \( H = \sum_\sigma \mathcal{H}_\sigma^{(1)} + \sum_{\sigma \sigma'} \mathcal{H}_\sigma^{(2)}, \)

\[
\mathcal{H}_\sigma^{(1)} = E_\sigma \sum_i \hat{n}_{\sigma i} + J_\sigma \sum_{\langle ij \rangle} \hat{a}_{\sigma i}^\dagger \hat{a}_{\sigma j},
\]

\[
\mathcal{H}_\sigma^{(2)} = U_{\sigma \sigma'} \sum_i \hat{n}_{\sigma i} \hat{n}_{\sigma' i} + \sum_{i \neq j} V_{\sigma \sigma'}(i-j) \hat{a}_{\sigma i} \hat{a}_{\sigma j}^\dagger + \sum_{\langle \langle ij \rangle \rangle} \sum_{\sigma''} T_{\sigma \sigma''}^{\sigma''}(i-j) \hat{a}_{\sigma i}^\dagger \hat{n}_{\sigma''} \hat{a}_{\sigma j}. \quad (2b)
\]

Parameters \( E_\sigma \) and \( J_\sigma \) come from the single-particle Hamiltonian and denote single-particle energy and nearest-neighbor tunneling in orbital \( \sigma \), respectively. The interparticle interaction has three contributions to the Hamiltonian (2b) as follows: (i) the onsite interaction energy of fermions occupying different orbitals \( \sigma \) and \( \sigma' \) of the same site \( U_{\sigma \sigma'} \), (ii) the long-range interaction energy of fermions occupying orbitals \( \sigma \) and \( \sigma' \) of different sites \( V_{\sigma \sigma'}(i-j) \), and (iii) the tunneling from orbital \( \sigma' \) at site \( j \) to the orbital \( \sigma \) at site \( i \) induced by the presence of an additional fermion at site \( i \) in orbital \( \sigma'' \) denoted by \( T_{\sigma \sigma''}^{\sigma''}(i-j) \). Without the loss of generality we can set \( E_\sigma = 0 \).

The Hamiltonian (2) is very general. To get a physical understanding of its properties, we start by examining the properties of density-density interactions. We calculate the interactions between a few of the lowest bands: \( s = \{000\}, p_s = \{010\}, p_x = \{001\}, p_{xz} = \{011\}, \) and \( p_y = \{011\} \). We find that the onsite interactions \( U_{s,p_s} = U_{s,p_x} \) are always repulsive. This means that putting two fermions in \( s \) or \( p_s \), or \( p_x \) orbitals simultaneously is energetically unfavorable. Remarkably, we find that \( U_{s,p_x} \) is always negative. This surprising attraction stems from the presence of the fermionic exchange term and the shape of the dipolar interactions (see Appendix A). The negativity of \( U_{s,p_x} \) cannot be solely explained by the attractive nature of the dipolar interaction separated along the \( z \) direction. It comes from the fact that the indistinguishable fermions occupy two orbitals in the same site. Therefore, it is not present in models with two different species as well as dipolar bosons. For higher order, we find that \( |U_{s,p_x}| > |U_{s,p_{xz}}| = |U_{s,p_y}| \). As \( U_{s,p_x} \) is negative, we look into the \( \kappa \)-\( D \) parameter space to investigate the breakdown of single-band approximation since two particles can occupy the same site. This behavior is controlled by the onsite energy cost \( \Delta = E_z + U_{s,p_x} \), where the energy gap between the \( s \) and \( p_x \) orbitals is given by \( E_z = \hbar \Omega \). In Fig. 1(a), for a fixed dipolar strength \( D = 8 \), we see that \( \Delta \) becomes negative when the lattice confinement \( \kappa \) goes below some critical value. In Fig. 1(b), we plot \( \Delta \) as a function of dipolar strength \( D \) and we see that, above some critical dipolar strength \( D \), \( \Delta \) becomes negative. This value of the critical \( D \) also decreases as the lattice confinement \( \kappa \) goes down. This also suggests that for low \( \kappa \), the single-band approximation will break down even for lower dipolar strength \( D \). In Fig. 2(a), we plot the various interaction strengths among different bands as a function of lattice confinement \( \kappa \). We note that, in addition for long-range interactions, we find that \( V_{s,s}(i) > V_{s,p_x}(i) > V_{p_x,p_x}(i) > 0 \). This results in the breaking of the single-band approximation due to presence of the \( p_x \) orbital. Thus, it is important to take into account at least the \( s \) and \( p_x \) orbitals to describe the dipolar fermions.

FIG. 1. (Color online) (a) We plot \( \Delta \) as a function of lattice confinement \( \kappa \) for different values of lattice depth \( V_0 \) and dipolar strength \( D = 8 \). The black solid line, red dashed line, and blue dash-dotted lines correspond to \( V_0 = 12E_x, 8E_x, \) and \( 6E_x \), respectively. (b) Here we plot \( \Delta \) as a function of dipolar strength \( D \) for different values of lattice confinement \( \kappa \) and lattice depth \( V_0 = 8E_x \). The black solid line, red dashed line, and blue dash-dotted lines correspond to \( \kappa = 9, 7, \) and \( 5 \), respectively.

FIG. 2. (Color online) Parameters of the Hamiltonian for \( V_0 = 8E_x \) as functions of the lattice confinement \( \kappa \). (a) Onsite interaction \(-U_{s,p_x}\) (blue line, -1-) and nearest-neighbor long-range interactions \( \frac{V_{s,s}(i)}{D^2}\) (solid black line, -2-), \( \frac{V_{s,p_x}(i)}{D^2}\) (dashed black line, -3-), and \( \frac{V_{p_x,p_x}(i)}{D^2}\) (dotted black line, -4-). The red line (-5-) shows the ratio \( V_{p_x,p_x}(2\epsilon_x)/T_{d1} \). (b) Magnitudes of the induced tunneling terms \( T_{p_x,p_x}(i,j)/D^4 \) (black solid line, -1-), \( T_{p_x,p_s}(i,j)/D^4 \) (red solid line, -2-), and \( T_{p_s,p_s}(i,j)/D^4 \) (blue solid line, -3-). The dashed black line (-4-) denotes the ratio \( T_{d1}/I_s \) for \( D = 10 \).
We next discuss the role of the interaction-induced tunnelings in Hamiltonian (2). Counterintuitively the most important contribution does not come from the induced tunneling in the $p_z$ band ($T^i_{p_z,p_z}(e_i)$) but from the interband tunneling which changes the $p_z$ orbital to the $p_x$ and $p_y$ ones ($T^i_{p_z,p_x}(e_i)$). Note that this interband tunneling is absent for usual single-particle tunneling as the Wannier states of different orbitals belongs to different Bloch bands [18]. From properties of particle tunneling as the Wannier states of different orbitals $\{ p_x, p_y, p_z \}$ and the tunneling. To find the lowest-energy classical crystal states we first look for the classical crystal states in the limit of $\hbar\Omega \gg \hbar\Delta$. From properties of interactions between orbitals $\sigma \in \{ s, p_z, p_x, p_y \}$. As we show later, the effect of $\{ p_x, p_y \}$ orbitals can result in strong long-range tunneling which is absent in the usual tight-binding models. Thus, the effect of orbitals higher in energy can, additionally, give rise to new processes which cannot be taken into account by renormalization of the parameters.

At this point let us note that our model shines a light on the various spin-Hamiltonian simulators created via polar molecules in optical lattices [5–7]. The standard way of treating such a systems is to rewrite a Hamiltonian as an effective spin model under the assumption that each lattice site can be occupied at most by one molecule. This simply means that one neglects the interaction between two indistinguishable particles at the same site. From our analysis, it follows that this assumption definitely breaks down when $\Delta = E_z + U_{s,p} < 0$, which is controlled by the dipolar strength $D$, optical lattice depth $V_0$, and trapping frequency $\Omega$, as plotted in Figs. 1(a) and 1(b). As one increases the lattice depth or dipolar strength, the trapping frequency in the $z$ direction $\Omega$ must be increased to prevent the breaking of the single-band approximation. This observation limits the parameter regimes accessible for the models used in Refs. [5–7].

### III. CLASSICAL GROUND STATE OF THE SYSTEM

To find the ground state of the system for a given filling, we first look for the classical crystal states in the limit of vanishing tunneling. Such an approach is valid as long as the excitation energies of the crystal state are much larger than the tunneling. To find the lowest-energy classical crystal states we follow the strategy described in Refs. [19,20]. Without the higher orbital effects, at most one fermion can occupy a given site. The corresponding Hamiltonian reads

$$\mathcal{H}_I = \sum_{i \neq j} V_{ss}(i-j)\hat{n}_i\hat{n}_j, \quad (3)$$

with the dipolar interaction in the $s$ band $V_{ss}(i-j) = V_{ss}/|i-j|^3$.

By taking the orbital effects into account, the corresponding Hamiltonian is defined in Eq. (2),

$$\mathcal{H}_{II} = E_\sigma \sum_i \hat{n}_{i\sigma} + \sum_\sigma U_{\sigma\sigma'} \sum_i \hat{n}_{i\sigma} \hat{n}_{i\sigma'},$$

$$+ \sum_{\sigma, \sigma'} \sum_{i \neq j} V_{\sigma\sigma'}(i-j)\hat{n}_{i\sigma}\hat{n}_{j\sigma'}, \quad (4)$$

where $\sigma$ denotes the $s$ and $p_z$ orbital fermions.

FIG. 3. (Color online) Pictorial diagram of the different checkerboard lattices for $n = 1/3$. The blue spheres denote $s$-orbital fermions and the smaller black spheres denote $p_z$ orbital fermions. (a) Ground-state crystal phase of Hamiltonian (3). (b) A 1/4 checkerboard lattice of $s$-band fermions and extra $p_z$ fermions with density $1/12$. (c) Density-wave structure of the effective bosons with filling $n_b = 1/6$ corresponding to the ground-state structure of the Hamiltonian (5). (d) The energies $E_{3s}$ (thick solid line), $E_{3b}$ (dashed line), and $E_{2c}$ (thin solid line) as functions of the trap frequency $\hbar\Omega/2E_R$ for dipolar strength $D = 10$.

We next consider the situation where each occupied site contains two fermions. In this case, we can define a corresponding hard-core bosonic operator at site $i$ as $b^\dagger_i = s_i^\dagger p_i^\dagger$ and $b_i = p_i s_i$, and the bosonic number operator $\hat{n}_b = b_i^\dagger b_i$. Subsequently, we can write an effective bosonic Hamiltonian as

$$\mathcal{H}_{III} = \Delta \sum_i \hat{n}_b^2 + \sum_{\sigma, \sigma'} \sum_{i \neq j} V_{\sigma\sigma'}(i-j)\hat{n}_i^\dagger \hat{n}_j^\dagger, \quad (5)$$

where $\sigma, \sigma' = s, p_z$, and $\Delta$ is the energy cost of having a composite boson as defined before. Equation (5) is similar to the bosonic dipolar system with a modified dipolar interaction and can simulate the crystal phases of bosonic bosons [21].

For concreteness, we first specifically choose $n = 1/3$. At filling $n = 1/3$ the ground state of the single-band Hamiltonian (3) forms a crystal structure in accordance with Ref. [8] and it is shown in Fig. 3(a). Its energy is $E_{3s}$. In the current paper, we analyze other structures as a ground state corresponding to the full Hamiltonian (4) from general arguments given in Refs. [19,20]. Two other structures can form the lowest-energy states and are presented in Figs. 3(b) and 3(c) with corresponding energies $E_{3b}$ and $E_{3c}$. In the 3b structure the $s$-band fermions form a 1/4 crystal structure and the remaining 1/12 $p$-orbital fermions occupy already-occupied sites. The third possible ground-state candidate Fig. 3(c) comes from the effective bosonic Hamiltonian (5) at filling $n_b = 1/6$. The energies of the three structures are plotted as functions of the harmonic trapping frequency for a dipolar strength $D = 10$ [Fig. 3(d)]. We find that the energy of the structure $3a$ is almost insensitive to the trapping frequency.
Therefore, such states are stable with respect to s-orbital tunneling. We also note that for the structure 3c, within the bosonic subspace, tunneling can arise in second-order processes and it is much lower than the binding energy of the bosons. The stability of structures like 3b with respect to p\(_z\)-orbital fermions will be discussed in the following sections.

IV. GROUND-STATE STRUCTURES NEAR \( n \gtrsim 1/4 \)

In this section we will look into the properties of the ground states near \( n = 1/4 \) filling. We show that the presence of higher orbitals not only changes the ground-state crystal structures, it also fundamentally changes the properties of such states. Specifically, we show that new forms of matter, like the smectic metal phase, can spontaneously form due to the effect of higher orbitals.

Here we consider the case when \( \Delta > 0 \) and, therefore, for low filling all fermions occupy only the s-orbital states. For filling \( n = 1/4 \) and large-enough \( D \geq 3 \) there is a nonvanishing single-particle excitation gap and the system is in the s-band insulator state [denoted by blue spheres in Fig. 5(a)] [8]. Situation changes dramatically for higher fillings. It can be simply understood using energy arguments.
The energy cost of putting additional fermions in the vacant site is given by $E_{\text{occ}} = V_{s,s}(e_1) + 2V_{s,s}(e_s + e_s) + \cdots$. In contrast, the cost of putting additional particles to the $p_z$ orbital of an occupied site is $E_{\text{occ}} = \Delta + 2V_{p_z,p_z}(2e_{s}) + \cdots$. For $D$ larger than some critical strength one finds that $E_{\text{occ}} < E_{\text{vac}}$. As an example, such conditions are fulfilled for $V_0 = 8E_R$, $D = 10$, and $\hbar\Omega \leq 14E_R$. Consequently, additional particles start to fill the $p_z$ band of previously occupied sites. In this scenario, energy-conserving dynamics of the system comes from the second-order processes involving tunneling to the next occupied site [along the $x$ direction in Fig. 5(a)]. To the leading order, this effective tunneling is given by

$$T^\parallel_{\text{eff}} \approx T^s_{\parallel,p_z}(e_s)^2/(|U_{s,p_z}| + E_{s}). \quad (6)$$

As $p_z$ fermions only exist with another $s$ fermion in the same site, the $p_z$ fermions can tunnel (with $T^\parallel_{\text{eff}}$) along one direction chosen by the insulator checkerboard geometry in $s$ band, in our case along $e_s$ [Fig. 5(a)]. Along $e_s$, on the other hand, there is no $s$-orbital fermion where a $p_z$ fermion can hop directly or by use of a second-order process without going to an higher energy state. Thus, the movement of the $p_z$ fermions is confined only to along $e_s$. Thus, the resulting system can be thought as stacks of one-dimensional chains or stripes placed along $e_s$ without interchain tunnelings [22]. The effective Hamiltonian governing the $p_z$ fermions can be written as $H_{\text{eff}} = T^\parallel_{\text{eff}} \sum_i \sum_{l,l'} c^\dagger_{l,i} c_{l',i} + H_{\text{intr}} + H_{\text{inter}}$, with intrachain Hamiltonian $H_{\text{intr}} = \sum_i \sum_{j,j'} V_{\text{intr}}(i,j)c_{i}^\dagger c_{j} + \text{H.c.}$ and interchain Hamiltonian $H_{\text{inter}} = \sum_{i,i'} \sum_{l,l'} V_{\text{inter}}(i,i')c_{i}^\dagger c_{i'} + \text{H.c.}$, where $c_{l,i}^\dagger$ and $c_{l,i}$ are creation and annihilation operators of $p_z$ fermions on the $s$-fermion-occupied site $i$ on chain $l$. The intrachain and the interchain interactions are given by $V_{\text{intr}}(i,j) = V_{p_z,p_z}((i - j)e_s)$ and $V_{\text{inter}}(i,i') = V_{p_z,p_z}((i - i')e_s + |l - l'|e_s)$, respectively.

The ground-state structure of this coupled-chain system is investigated by introducing the bosonized fields $\phi_{l,R/L}$ related to the Fermi operator $c_{l,i}$ rewritten in the continuum limit as $c_{l,i} \rightarrow \Psi_{l}(x) + \Psi_{R}(x)$ [23,24]. Near the left and right Fermi momenta $\pm k_1$, we can write $\Psi_{l,R/L}(x) = F_{R/L} \exp[\pm ik_1 x - i\phi_{l,R/L}(x)]/\sqrt{2\pi}\epsilon$, where $\epsilon$ is a cutoff parameter and $F_{R/L}$ are Klein factors. The Fermi moment is given by the density of $p_z$ fermions which, in terms of total density $n$, reads $k_1 \approx (4n - 1)\pi$. By writing the bosonized phase field $\theta_{l}(x) = \phi_{l,R}(x) - \phi_{l,R}(x)$ of $
abla \theta_{l}(x)/\sqrt{2\pi}\epsilon$ in terms of its Fourier transform $\theta_{l}(x)$ along $e_s$, the Lagrangian for the system reads

$$\mathcal{L} = \int_{-\pi}^{\pi} \frac{dq}{2\pi} \frac{1}{2} \left( \frac{\partial \theta_{l}(q)}{\partial t} \right)^2 - v(q_{s}) \left( \frac{\partial \theta_{l}(q)}{\partial x} \right)^2. \quad (7)$$

The interaction parameter $K(q_{s})$ and sound velocity $v(q_{s})$ are determined by the details of the dipolar interactions (see Appendix C).

Interchain interactions induce additional charge-density wave (CDW) perturbation, $\mathcal{L}_{\text{CDW}} = \mathcal{L}_{\text{CDW},1} + \mathcal{L}_{\text{CDW},2} + \cdots,$ with

$$\mathcal{L}_{\text{CDW},1} = \frac{1}{u} \sum_{i} V_{p_z,p_z}((2N + 1)e_s + 2e_{s}) \cos[(2N + 1)\theta_{l}]/(\theta_{l} - \theta_{l+1})$. \quad (8a)$$

$$\mathcal{L}_{\text{CDW},2} = \frac{1}{u} \sum_{i} V_{p_z,p_z}((2N + 1)e_s + 4e_{s}) \cos[(2N)\theta_{l}]/(\theta_{l} - \theta_{l+2})$. \quad (8b)$$

Here $\mathcal{L}_{\text{CDW},1}$ is the CDW coupling between two chains with distance $L = 1,2,\ldots$. Consequently, following Ref. [25], the scaling dimension of the CDW operator is given by $\eta_L = 2\int_{-\pi/2}^{\pi/2} \frac{1}{\epsilon^2} \cos L q_{s} d\epsilon$. When $\eta_L > 2$ for all $L$, the CDW operator is irrelevant. The stable phase then has properties similar to a 1D Luttinger liquid with low-energy bosonic collective excitations. This state preserves the smectic symmetry $\theta_l + \alpha_l$, with $\alpha_l$ constant on each chain. This phase is known as the smectic-metal phase [23] as there is metallic behavior along the chain with an insulating density wave order along the transverse direction. This phase is a peculiar example of the spontaneous emergence of non-Fermi liquid behavior in two-dimensional Fermi systems. In contrast, when $\eta_L < 2$, then $p_z$ fermions becomes unstable towards formation of stripe crystals. In Fig. 5(b) we plot $\eta_1$ and $\eta_2$ as functions of total density $n$ for $D = 10$ and $\hbar\Omega = 14E_R$. It is clear that for $1/4 < n < n_c$ there is a smectic-metallic phase while, for $n > n_c$, the system goes to a stripe-crystal phase. We have checked $\eta_1$ to $L = 4$ and all reside between $\eta_1$ and $\eta_2$.

### V. GROUND-STATE STRUCTURES NEAR $n \geq 1/2$

In this section let us discuss the case of filling $n = 1/2$ where, for low dipolar strength $D$, due to the same reasons as before, fermions will occupy only the $s$ band and the ground state of the system is the checkerboard insulator (see Appendix C), as denoted by the solid blue and open red spheres in Fig. 5(b). To look for properties of the system with additional particles, we define the deviation from half-filling $\delta n = n - 1/2$ and we introduce the corresponding chemical potential $\mu(\delta n)$. From energy arguments we find that two scenarios can occur. The additional fermion (i) occupies a vacant site with energy cost $E_{\text{vac}} = 4V_{s,s}(e_s) + 8V_{s,s}(2e_s + e_s) + \cdots$ or (ii) it goes to the $p_z$ orbital of an occupied site with energy cost $E_{\text{occ}} = \Delta + 4V_{s,p_z}(e_s + e_s) + V_{s,p_z}(2e_s) + \cdots$. Consequently, in the second scenario (when $E_{\text{occ}} \leq E_{\text{vac}}$), all extra fermions will occupy the $p_z$ orbitals of the already occupied sites. As an example, such conditions are fulfilled for $V_0 = 8E_R$, $D = 8$, and $\hbar\Omega \leq 10E_R$. In such a case, $\delta n$ corresponds to the filling of $p_z$ band fermions. The parallel tunneling of the $p_z$ fermions between the occupied sites will again arise from the second-order processes (6). Moreover, tunneling to the diagonally occupied site $T^\parallel_{\text{eff}} \approx -(J_t - T_\perp^{\parallel})^2/|U_{s,p_z}|$, for $D \sim 8$ it is 400 times smaller than $T^\parallel_{\text{eff}}$. Consequently, fermions in the $p_z$ orbitals can move in independent square sublattices [either the thick red or blue sublattice shown in the Fig. 5(b)]. Note that fermions cannot tunnel between different sublattices. Thus, we can
describe the system of the $p_z$ fermions in the blue (thick red) lattice as pseudo-spin-up (-down). By introducing operators $\tilde{c}_{is}$, where $s \in \{\uparrow, \downarrow\}$, the effective Hamiltonian can be written as $H_{\text{eff}} = T_1 \sum_{s} \sum_{<ij>} \tilde{c}_{i}s \tilde{c}_{j}s + H_{\text{int}}$ with

$$H_{\text{int}} = V_{\uparrow\uparrow} \sum_{s} \sum_{<ij>} \tilde{n}_{is} \tilde{n}_{js} + V_{\uparrow\downarrow} \sum_{s} \tilde{n}_{is} \tilde{n}_{js},$$  \hspace{1cm} (9)$$

where $\tilde{n}_{is} = \tilde{c}_{is}^{\dagger} \tilde{c}_{is}$. For convenience we introduce $V_{\uparrow\uparrow} = V_{p_{z}p_{z}}(2\epsilon)$ and $V_{\uparrow\downarrow} = V_{p_{z}p_{z}}(\epsilon_{+} + \epsilon_{-})$. Note that now $[,]$ is understood as a nearest-neighbor in a given sublattice. Nearest neighbors between different sublattices is denoted by $[,]$. The modified lattice constant of the sublattices is $\tilde{a} = 2a$. In this way we are able to study the system properties with the weak-coupling theory. We investigate the emergence of triplet superconductivity between the same pseudospin fermions, arising via the KL mechanism [18] (magnetic instabilities are discussed in Appendix D). We look for Cooper pairs with chiral $p$-wave symmetry. The effective interaction between fermions in KL mechanism in terms of the scattering momentum $k - k' = q$ can be written as

$$V_{\text{eff},s,s}(q) = V_{\uparrow\uparrow} \eta_{q} \sum_{p} \left[ (V_{\uparrow\uparrow}^{2} q_{p}^{2} + V_{\uparrow\downarrow}^{2} \beta_{q}^{2} Q_{q,p} ight. - 2V_{\uparrow\uparrow}^{2} q_{k-p} Q_{q,p} - V_{\uparrow\uparrow}^{2} \eta_{k-p}^{2} k_{p}^{2} k_{p}^{2} Q_{q,p}^{2} + V_{\uparrow\downarrow}^{2} \eta_{k-p}^{2} k_{p}^{2} k_{p}^{2} Q_{q,p}^{2} \right],$$  \hspace{1cm} (10)$$

where $Q_{q,p} = f(\epsilon_{p} - f(\epsilon_{p} + q)), f(\epsilon)$ is the Fermi distribution function, $\epsilon_{p} = 2T_{\text{eff}}^{2}[\cos(q_{x}\tilde{a}) + \cos(q_{y}\tilde{a})]$ is the dispersion and $\eta_{q} = [\cos(q_{x}\tilde{a}) + \cos(q_{y}\tilde{a})] + \beta_{q} = 4[\cos(q_{x}\tilde{a}/2) \cos(q_{y}\tilde{a}/2)].$ The summation in (10) comes from taking into account the second-order terms represented by diagrams shown in Fig. 6(a). The two terms inside the first bracket in (10) comes from the top-left diagram in Fig. 6(a), while the next two terms comes from the top-right and bottom-left diagrams representing vertex corrections. The last term in (10) comes from the bottom-right diagram in Fig. 6(a) denoting exchange interactions. By performing the integration over the momentum in the limit of $T \rightarrow 0$, we finally get antisymmetric part of effective coupling $\{V_{\text{eff}}(q)\} = -\lambda(T,\mu)[\sin(k_{x}\tilde{a})\sin(k_{x}'\tilde{a}) + \sin(k_{y}\tilde{a})\sin(k_{y}'\tilde{a})]$, where $\lambda(T,\mu) = 2V_{\uparrow\uparrow} + \frac{V_{\uparrow\uparrow}^{2}}{\pi T_{\text{eff}}} F_{1}(T,\mu) - \frac{V_{\uparrow\downarrow}^{2}}{\pi T_{\text{eff}}} F_{2}(T,\mu)$. Functions $F_{1}$ and $F_{2}$ originate in the second-order corrections and their detailed forms are given in Appendix E. The point is that, due to the Van Hove singularity in density of states, function $F_{2}$ contains a logarithmic divergence. At the same time, function $F_{1}$ is analytical due to the dressing of the density of states. This means that there always exists a finite critical $\mu$ above which the interaction is attractive and superfluidity appears. From the BCS theory one can get an estimate of the transition temperature $T_{c}$ (derivation is shown in Appendix E). In Fig. 6(b) we plot the transition temperature $T_{c}$ as a function of the deviation $\delta n$, for example, the parameters discussed previously. For $\delta n \sim 0.22$ we get $T_{c} \sim 0.2 J_{\perp} (\sim 1$ nK). Thus, the ground state has a checkerboard density pattern due to the fermions in the $s$ orbital and $p$-wave superfluid fermions in the $p_z$ orbital at temperature below $T_{c}$. This can be considered an exotic supersolid. In contrast to the two-species models, $s$ and $p_z$ fermions are indistinguishable and, therefore, superfluid and CBW orders are not independent. This comes from the fact that onsite coupling $U_{pp}$ includes the exchange term between fermions in the $s$ and $p_z$ orbitals. Such a term is not present in standard two-species models.

VI. CONCLUSIONS

In conclusion, we have derived a generalized Hubbard model for dipolar fermions in an optical lattice by taking into account higher orbitals. One important aspect of our findings is that the single-orbital Hubbard models of dipolar fermions as used in literature breaks down even for moderate dipole strengths and realistic trapping potentials. Furthermore, we have shown that the effect of these higher orbitals leads to interesting phenomena. Due to the strong interaction-dependent hopping terms in higher orbitals, these systems can be described by effective weakly interacting theories. For particular parameters, near $n \geq 1/4$, we found a crossover to the one-dimensional physics resulting in simultaneous metallic and density wave properties. For $J_{\perp} \ll V_{\text{eff}}(\epsilon_{s})$, the $s$ fermion checkerboard order is given by the configuration in Fig. 2(a). As $J_{\perp}$ is increased there will be single-particle and dipole excitations at different regions of the $n = 1/4$ checkerboard crystal similar to the one considered in Ref. [26] for Wigner-Hubbard crystals due to Coulomb interaction. These excitations can induce interchain tunneling at different regions and the resulting model will be subject of future study. For other set of parameters, $n \geq 1/2$, the system can be described by a weakly interacting Hubbard model with pseudospin originating from the lattice geometry. Using the KL theory, we found a transition to the chiral $p$-wave superfluidity due to the $p_z$ fermions without destroying the checkerboard order created by the $s$ fermions.

The parameters used here are experimentally realistic and for some parameters are even currently accessible. Nevertheless, let us note that to apply our theory for real experimental scenario one should take into account losses induced by reactivity of the molecules. In our paper we completely neglect this effect by assuming that the loss rates are much smaller than the considered tunneling processes. In fact, this is true even for highly reactive molecules like KRB. As reported in Ref. [27]
for a lattice depth of ~25 recoil energy (\( \Omega \sim 5 \)) the loss rate in the \( p \)-orbital \( v_{0} \) is of the order of \( 5 \times 10^{4} \) s\(^{-1} \). In comparison, the interaction-induced interorbital tunneling rate \( T'_{p_z,p_z}(-e_{x}) \) is in the order of \( 5 \times 10^{2} \) s\(^{-1} \) for \( D = 8 \) and \( \Omega = 5 \). It means that the ratio \( \gamma/T'_{p_z,p_z}(-e_{x}) \sim 1 \), which can make the system stable with respect to such tunneling processes.

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APPENDIX A: DERIVATION OF THE PARAMETERS \( U_{s,p_{z}}, T_{\sigma p_{z}p_{z}}^{s}, \) AND \( T'_{\sigma p_{z}p_{z}} \)

In this section we represent the onsite interaction term \( U_{s,p_{z}} \) and interaction-induced hopping terms \( T_{\sigma p_{z}p_{z}}^{s} \) and \( T'_{\sigma p_{z}p_{z}} \) in terms of the single-particle wave function \( \tilde{w}_{i}(r) \) in orbital \( \sigma \) localized on site \( i \). The orbital index \( \sigma = \{pm\} \), denoting \( p_{m} \) and \( l \) excitations in the \( x, y, \) and \( z \) directions, respectively. The \( s \) orbitals then can be written as \( \tilde{w}_{i\sigma}(r) = w_{\sigma}(x)w_{\sigma}(y)\phi_{\sigma}(z), \) where \( w_{\sigma}(x), w_{\sigma}(y) \) are the lowest-band one-dimensional Wannier functions and \( \phi_{\sigma}(z) \) is the ground-state wave function of the harmonic oscillator in the \( z \) direction. Similarly, we can write \( \tilde{w}_{i\sigma}(r) = w_{\sigma}(x)w_{\sigma}(y)\phi_{\sigma}(z) \) and \( \tilde{w}_{i\sigma}(r) = w_{\sigma}(x)w_{\sigma}(y)\phi_{\sigma}(z). \) Here \( w_{\sigma}(x) \) is the Wannier functions in the first band and \( \phi_{\sigma}(z) \) is the first excited state of the harmonic oscillator in the \( z \) direction. For simplicity we took \( j = 1 + e_{x}. \) From this we can write various parameters as

\[
U_{s,p_{z}} = \int \left\{ w_{\sigma}(x)w_{\sigma}(y) \right\}^{2} \left\{ w_{\sigma}(x')w_{\sigma}(y') \right\}^{2} \times \Phi_{\sigma}(z,z') \sqrt{(r-r')d\rho d\rho'}, \quad (A1a)
\]

\[
T_{\sigma p_{z}p_{z}}^{s}(e_{x}) = \int w_{\sigma}(x)w_{\sigma}(y)w_{\sigma}(x')w_{\sigma}(y') \times \Phi_{\sigma}(z,z') \sqrt{(r-r')d\rho d\rho'}, \quad (A1b)
\]

\[
T'_{\sigma p_{z}p_{z}}(e_{x}) = \int w_{\sigma}(x)w_{\sigma}(y)w_{\sigma}(x')w_{\sigma}(y') \times \Phi_{\sigma}(z,z') \sqrt{(r-r')d\rho d\rho'}, \quad (A1c)
\]

\[
\Phi_{\sigma}(z,z') = (\phi_{\sigma}(z))^{2}(\phi_{\sigma}(z')^{2} - \phi_{\sigma}(z')\phi_{\sigma}(z)). \quad (A1d)
\]

The integrations over \( z, z' \) can be done analytically using convolution theorem in the momentum space. Consequently, in the momentum space we get

\[
V(k_{\perp}) = \mathcal{F} \left\{ \int \Phi_{\sigma}(z,z') \sqrt{(r-r')d\rho d\rho'} \right\} = 2\sqrt{2\pi}D l_{c} \left\{ (kl_{z})^{2} - \frac{\sqrt{\pi}}{2} kl_{z}(1+ (kl_{z})^{2}) \right\} \mathrm{erfcx} \left( \frac{kl_{z}}{\sqrt{2}} \right), \quad (A2)
\]

\[
\mathcal{L}_{\text{intra}} = n \int_{-\pi}^{\pi} dq_{x} K_{0} \left[ \left( \frac{\partial \theta_{q}(x)}{\partial t} \right)^{2} - \left( \frac{\partial \theta_{q}(x)}{\partial x} \right)^{2} \right], \quad (B2a)
\]

![FIG. 7.](Color online) (a) Pictorial diagram for \( n > 1/4 \). The blue spheres denote \( s \)-orbital fermions and the smaller black spheres denote \( p_{z} \) orbital fermions. The \( s \)-orbital fermions constitute the underlying \( 1/4 \) checkerboard structure. The \( p_{z} \) fermions move with the effective tunneling \( T_{\text{eff}}(\text{arrow 1}) \) only along the shaded regions. Interaction between neighboring \( p \)-band fermions is equal to \( V_{p_{z}p_{z}}(2e_{x}) \) (arrow 2). (b) Density-wave structure at filling \( \pi n = 1/4 \). The dark and light shadings denote higher and lower density of the \( p_{z} \) fermions, respectively.

APPENDIX B: LUTTINGER LIQUID DESCRIPTION FOR \( n > 1/4 \)

As we explained in the main text, for filling \( n > 1/4 \) and parameters \( V_{0} = 8E_{R}, D = 10, \) and \( \hbar \Omega \sim 14E_{R} \), the ground-state structure is given by a \( 1/4 \) checkerboard structure formed by \( s \) fermions. The \( p_{z} \) fermions (with density \( 4n - 1 \)) move in the occupied sites along the \( x \) direction [Fig. 7(a)]. The resulting system can be thought as stacks of one-dimensional chains or stripes placed along \( x \) without interchain tunnelings. The effective Hamiltonian governing the \( p_{z} \) fermions can be written as (see the main text)

\[
H_{\text{1D}} = T_{\text{eff}} \sum_{i} \left\{ \sum_{\langle j \rangle} \hat{c}_{i\sigma}^{\dagger} \hat{c}_{j\sigma} + H_{\text{intra}} + H_{\text{inter}} \right\}, \quad (B1a)
\]

\[
H_{\text{intra}} = \sum_{i} \sum_{\langle j \rangle} V_{\text{intra}}(i,j) \hat{c}_{i\sigma}^{\dagger} \hat{c}_{j\sigma}, \quad (B1b)
\]

\[
H_{\text{inter}} = \sum_{i,j} \sum_{\langle i',j' \rangle} V_{\text{inter}}(i,i') \hat{c}_{i\sigma}^{\dagger} \hat{c}_{j'\sigma}^{\dagger} \hat{c}_{i'\sigma} \hat{c}_{j\sigma}. \quad (B1c)
\]

The bosonized form of the intrachain Lagrangian reads
where the Luttinger liquid parameter

\[ K_0 = \left[ \frac{2\pi T_{\text{eff}}}{2\pi T_{\text{eff}}^2} \sin \theta_1 + \left[ \sum_{\mu} \left( 2 - \cos 2\theta_1 \right) - \cos 2\theta_1 \right] \right]^{1/2} \]

and the sound velocity

\[ u^2 = \left( \frac{2\pi T_{\text{eff}}}{2\pi T_{\text{eff}}^2} \sin \theta_1 + \left[ \sum_{\mu} \left( 2 - \cos 2\theta_1 \right) - \cos 2\theta_1 \right] \right)^2 - \left[ \sum_{\mu} \left( 2 - \cos 2\theta_1 \right) \right] (1 - \cos 2\theta_1)^2. \] (B2b)

and sound velocity \( v(q_1) = K(q_1)/K_0 \). Interchain interactions induce an additional CDW perturbation, \( \mathcal{L}_{\text{CDW}} = \mathcal{L}_{\text{CDW,1}} + \mathcal{L}_{\text{CDW,2}} + \cdots \) with

\[
\mathcal{L}_{\text{CDW,1}} = \frac{1}{\mu} \sum_N V_{p_p,p} \left[ 2N + 1 \right] e_s + 2e_s \cos \left[ 2N + 1 \right] \theta_1 \]

\[
\times \cos \left[ 2\sqrt{\pi} \left( \theta_1 - \theta_{i-1} \right) \right]. \] (B5a)

\[
\mathcal{L}_{\text{CDW,2}} = \frac{1}{\mu} \sum_N V_{p_p,p} \left[ 2N e_s + 4e_s \right] \cos \left[ 2N \theta_1 \right] \]

\[
\times \cos \left[ 2\sqrt{\pi} \left( \theta_1 - \theta_{i+2} \right) \right]. \] (B5b)

At half-filling, i.e., \( \theta_1 = \pi/2 \), we see that \( \mathcal{L}_{\text{CDW,1}} = 0 \). It means that the charge-density wave instability is induced by the nearest-neighbor interchain interaction. We checked that this interaction is much weaker than the tunneling \( T_{\text{eff}} \). It means that the smectic-metal phase discussed in the paper will be stable to low-enough temperatures.

**APPENDIX C: GROUND-STATE STRUCTURE OF s-ORBITAL FERMIONS AT N = 1/2**

To look into the ground state of s-orbital fermions at \( n = 1/2 \), we express the average density \( \langle \rho_{\alpha,s} \rangle = (1 + (-1)^{\uparrow +} \delta)/2 \), where \( \delta \) is the order parameter. We also define the single-particle imaginary time Green functions \( \tilde{G}(t - j, \tau) = \langle T \hat{a}_{\alpha,s}(\tau) \hat{a}_{\beta,s}(0) \rangle \), where \( T \) denotes time ordering. By following the description referred in Refs. [28,29] we find the following equations for \( G \) in the momentum space:

\[
[\omega + \mu - 2V(1 - \delta')]G_1(k, \omega) = \epsilon_k G_2(k, \omega), \]

\[
[\omega + \mu - 2V(1 + \delta')]G_2(k, \omega) = \epsilon_k G_1(k, \omega). \]

where the kinetic energy \( \epsilon_k = 2J_s \cos k + \cos k' \), the effective potential \( V = \sum_{\mu} V_{\mu}(\mu) \), and \( \delta' = 2\left( \sum_{i \neq \mu} \epsilon_v(i) - \sum_\mu \epsilon_v(\mu) \right)/V \). In the position space \( G_1(i) \) \( (G_2(i)) \) is equal to \( G(i) \) for \( i = i \), even (odd). These mean-field equations for \( G_1 \) and \( G_2 \) are similar to the ones found for the extended Hubbard model, with

We next include the bosonized form of the interchain Hamiltonian which results in the total Lagrangian \( \mathcal{L}_{1D} = \mathcal{L} + \mathcal{L}_{\text{CDW}} \), where

\[
\mathcal{L} = u \int_{-\pi}^{\pi} \frac{dq_1}{2\pi} \left( K(q_1) \left( \frac{1}{v(q_1)} \left( \frac{\partial \theta_{q_1}}{\partial t} \right)^2 - v(q_1) \left( \frac{\partial \theta_{q_1}}{\partial x} \right)^2 \right) \right). \] (B3)

Here the modified Luttinger parameter is given by

\[
\frac{K(q_1)}{K_0} = \left[ 1 + \frac{4 \sum_{\mu} \left[ \sum_{\mu} \left( 2 - \cos 2\theta_1 \right) - \cos 2\theta_1 \right] (2 - \cos 2\theta_1) \right]^{1/2}. \] (B4)

and a renormalized nearest-neighbor interaction and density imbalance [28,29]. Then, by solving equations (C1), we find that, in the strong coupling limit, \( \delta = 1 - \frac{3\beta_2}{2\pi v_1(\epsilon_0)} (\delta \approx 0.98 \) for \( D = 8 \) and \( h\Omega = 10E_R \)). Thus, our assumption of a checkerboard lattice with alternative sites occupied is justified.

**APPENDIX D: TRANSITION TEMPERATURE FOR STONER FERROMAGNETISM AND CHARGE-DENSITY WAVE INSTABILITY OF THE p_s FERMIONS FOR n > 1/2**

In this section we discuss the appearance of Stoner ferromagnetism and CDW instability of the \( p_s \) fermions. To do this we transform to momentum space and introduce charge fluctuations \( \rho_q = \sum_{i} c_{k+q,i} c_{k,i} \) and spin fluctuations \( S_q = \sum_{s} c_{k+q,s} c_{k,s} \) operators \( (s \in \{ \uparrow, \downarrow \}) \). Subsequently, we rewrite Eq. (4) from the main text in the momentum space as

\[
H_{\text{int}} = \frac{1}{4} \sum_q \left( \sum_{\mu} (2V_{\mu} \eta_q + V_{\mu} \beta_q) \rho_q \rho_{-q} \right)
+ \frac{1}{4} \sum_q \left( \sum_{\mu} (2V_{\mu} \eta_q - V_{\mu} \beta_q) S_q S_{-q} \right) \] (D1)

where \( \eta_q = 2[\cos(q_1 \alpha) + \cos(q_1 \beta)] \) and \( \beta_q = 4[\cos(q_1 \alpha)/2] \cos(q_1 \alpha/2) \). The system with interactions described by (D1) can manifest three possible magnetic instabilities: CDW, spin-density wave (SDW), and ferromagnetic instability. At \( \delta n = 1/4 \) (each sublattice is half-filled with \( p_s \)-orbital fermions), as \( \beta_q = 0 \) for nesting vector \( q = (\pm \pi, \pm \pi) \), interaction in the spin channel becomes repulsive and, therefore, SDW order is absent. In the spin channel, the onset of an Stoner ferromagnetism is given by the divergence of susceptibility with momentum \( q = (0, 0) \). This condition can be written as \( \lambda_\alpha(0) = 1 \), where \( \lambda_\alpha = \frac{1}{2} \left( V_{\mu} \beta_q - 2V_{\mu} \eta_q \right) \), and the bare susceptibility \( \chi(0) = \lim_{\eta \to 0} \int dp Q_{q,p} \) with

\[
Q_{q,p} = \frac{\left( \epsilon_p - \epsilon_q \right)^2}{\epsilon_p - \epsilon_q} \]

Here \( \epsilon(q) \) is the Fermi distribution function and \( \epsilon_p = 2T_{\text{eff}} \cos^2(q_1 \alpha) + \cos^2(q_1 \beta) \) is the
density of states we get. Consequently, in the limit of \( T \to 0 \),
\[ N(\epsilon) = K \sqrt{1 - (\epsilon + \mu)^2/16T_{\text{eff}}^2}/2\pi^2T_{\text{eff}}, \]
with \( K(.) \) being an elliptic integral of first kind. Substituting the
density of states we get \( \chi(0) \approx N(T) \). As \( \mu \to 0 \), or density of \( p_\uparrow \) fermions are near \( 1/4 \), due to the logarithmic divergence of \( K \), the transition temperature for the Stoner ferromagnetism is given by
\[ T_{\text{st}} \approx 8T_{\text{eff}}^{\uparrow\uparrow} \exp \left[ -\frac{2\pi^2 T_{\text{eff}}^{\uparrow\uparrow}}{\lambda_{\text{st}}} \right]. \]
The case \( V_0 = 8E_R \), around \( D \approx 8 \), \( \lambda_{\text{st}}/2\pi^2 T_{\text{eff}}^{\uparrow\uparrow} \approx 0.1 \), corresponds to a very low Stoner temperature \( T_{\text{st}} \approx 10^{-4}T_{\text{eff}} \).

We next discuss the checkerboard charge-density wave structure due to the \( p_\uparrow \) fermions at \( \delta n = 1/4 \). Due to nesting, each fermionic component will be unstable towards CDW. In the density channel, the onset of an CDW is indicated by the divergence of susceptibility with momentum \( \mathbf{q} = (\pi,\pi) \). The condition for \( p_\uparrow \) fermion CDW can be written as
\[ \lambda_{\text{CDW}}(\pi,\pi) = -1, \]
and the bare susceptibility
\[ \chi(\pi,\pi) = \lim_{q \to (\pi,\pi),\omega \to T} \int d\mathbf{p} \frac{f(\epsilon_p) - f(\epsilon_{p-q})}{\omega - \epsilon_p + \epsilon_p} F(T,\mu) \]
\[ \approx (\ln(8T_{\text{eff}}^{\uparrow\uparrow}/T))^2/2\pi^2T_{\text{eff}}^{\uparrow\uparrow}. \]

Subsequently, the transition temperature to the CDW is given by
\[ T_{\text{CDW}} \approx 8T_{\text{eff}}^{\uparrow\uparrow}(\pi,\pi) \approx 0.35J_0(\sim 2\text{ nK}) \text{.} \] One should note that due to the relative shift of sublattices the resulting density modulation in this phase looks like stripes rather than the standard checkerboard structure as shown in Fig. 7(b).

### APPENDIX E: DERIVATION OF EFFECTIVE INTERACTION IN THE TRIPLET CHANNEL

In this section we derive the \( p \)-wave interaction from the Kohn-Luttinger effective interaction in terms of the scattering momentum \( \mathbf{q} = \mathbf{k} - \mathbf{k}' \). For this purpose we rewrite Eq. (6) from the original paper
\[ V_{\text{eff},s}(q) = V_{\uparrow\uparrow}\eta_q - \sum_p \left( \left( V_{\uparrow\downarrow}^2 n_{q\uparrow}^2 + V_{\uparrow\downarrow}^2 \beta_{\uparrow\downarrow}^2 \right) Q_{q,p} \right) \]
\[ - 2V_{\uparrow\downarrow}^2 \eta_{k-p} Q_{p,q} - V_{\uparrow\downarrow}^2 \eta_{k'-p} \eta_{k-k'} Q_{p,k-k',p} \] (E1)
where \( Q_{p,q} = \frac{f(\epsilon(p)) - f(\epsilon(p-q))}{\epsilon(p) - \epsilon(p-q)} \) with \( f(.) \) being the Fermi distribution function.

First, we put the expression of \( \eta_q = 2[\cos(q_\uparrow \tilde{a}) + \cos(q_\downarrow \tilde{a})] \) and \( \beta_q = 4[\cos(q_\uparrow \tilde{b})/2 \cos(q_\downarrow \tilde{b})/2] \) back to (E1). As we are interested in \( p \)-wave interaction, after expanding (E1) in terms of the momenta \( k_\uparrow, k_\downarrow, k' \), we keep terms proportional to \( \sin k_\uparrow \tilde{a} \sin k'_\downarrow \tilde{a} + \sin k_\downarrow \tilde{a} \sin k'_\uparrow \tilde{a} \). In this way we get
\[ \{V_{\text{eff}}(q)\} = 2V_{\uparrow\downarrow} - \sum_p Q_{q,p} \left( 4(V_{\uparrow\downarrow})^2 - 8(V_{\uparrow\downarrow})^2 \right) \]
\[ \times \left( \cos(k_\uparrow \tilde{a} - p_\downarrow \tilde{a}) + \cos(k_\downarrow \tilde{a} - p_\uparrow \tilde{a}) \right) \]
\[ + 4(V_{\uparrow\downarrow})^2 \sum_p Q_{q,p} \left( \sin k_\uparrow \tilde{a} \sin k'_\downarrow \tilde{a} \sin^2 p_\downarrow \tilde{a} \right) \]
\[ + \sin k_\downarrow \tilde{a} \sin k'_\uparrow \tilde{a} \sin^2 p_\uparrow \tilde{a}, \]
(E2)
where \( q = k - k' \). By converting sums to integrals in (E2) we have to compute terms of the form \( \int d\mathbf{p} Q_{q,p} \int d\mathbf{p} \sin^2 p_\downarrow \alpha \beta_{q,p} \), and \( \int d\mathbf{p} \cos p_\alpha \alpha_0 \beta_{q,p} \). In the limit of vanishing temperature \( T \to 0 \) we approximate all integrals \( \int d\mathbf{p} G(\mathbf{p}) \equiv \int d\mathbf{p} \approx \int d\mathbf{p} G(\mathbf{p}) \approx \int N_{\text{eff}}(\epsilon,\mu) d\epsilon \) for arbitrary function \( G(\mathbf{p}) \). The effective density of states reads \( N_{\text{eff}}(\epsilon,\mu) = \int d\mathbf{K} G(k) \delta(\epsilon - \epsilon_k) \). We see that the only first term inside the third bracket in (E2) is not dressed by \( \cos(p_\alpha \tilde{a}) \) or \( \sin(p_\alpha \tilde{a}) \). Hence, the effective density of states for this term contains Van Hove singularity. All other terms with in (E2), due to the dressing by \( \cos(p_\alpha \tilde{a}) \) or \( \sin(p_\alpha \tilde{a}) \), are analytic. For convenience, we reexpress \( \{V_{\text{eff}}(q)\}_q \sim -\lambda(T,\mu)\sin(k_\uparrow \tilde{a})\sin(k'_\downarrow \tilde{a}) + \sin(k_\downarrow \tilde{a})\sin(k'_\uparrow \tilde{a}) \), where \( \lambda(T,\mu) = 2V_{\uparrow\downarrow} + \frac{V_{\uparrow\downarrow}^2}{\pi T_{\text{eff}}} F_1(T,\mu) - \frac{V_{\uparrow\downarrow}^2}{\pi T_{\text{eff}}} F_2(T,\mu) \). The functions \( F_1 \) and \( F_2 \) are given by
\[ F_1 = \frac{4}{\pi} F(1 - (T + |\mu|)^2)/(4T_{\text{eff}}^2), \]
\[ F_2 = \frac{2}{\pi} K[1 - (T + |\mu|)^2/(4T_{\text{eff}}^2), \]
where \( F(x) = E(x) - (1 - x)K(x) \) with \( K(.) \) being the elliptic integral of the first kind and \( E(.) \) is the elliptic integral of the second kind. Then we can write the BCS equation for the transition temperature \( T_c \) as [30]
\[ V_p N_{\text{eff}}(0,\mu) \log \left( 1 - \left[ \frac{\mu}{4T_{\text{eff}}^2} \right]^{1/2} \right)^{1/2} \right]^{1/2} \]
\[ = 1, \]
(E4)
where the effective density of state is given by
\[ N_{\text{eff}}(\epsilon,\mu) = \sum_k \delta(\epsilon + \mu - \epsilon_k) \sin^2(k_\alpha \alpha_0) \]
\[ \approx F[1 - (\epsilon + |\mu|)^2/(4T_{\text{eff}}^2)]/\pi^2 T_{\text{eff}}. \]
(E5)
As discussed in the main text, the ground state has a checkerboard density pattern due to the \( s \) fermions and \( p \)-wave superfluid \( p_\uparrow \) fermions at temperature below \( T_c \). From previous section, we see that at \( \Delta n = 1/4 \) (half-filled \( p_\uparrow \) fermions for the red and blue lattices), the transition temperatures for the \( p_\uparrow \) fermion CDW and \( p_\uparrow \) fermion superfluidity are similar. This will result in a competition or coexistence between both instabilities for the \( p_\uparrow \) orbital fermions. A detailed account of such scenario is beyond the scope of this paper.
[22] For our parameters, there can be localized excitations due to s-fermion tunneling with probability \( \exp(-0.3V_{ss}(e_i)/J_s) \approx 0.01 \). Such excitations can couple two chains, but due to the small probability of such excitations we will neglect them.