Quantum entanglement is a fascinating phenomenon, especially if it is observed at the macroscopic scale. Importantly, macroscopic quantum correlations can be revealed only by accurate measurement outcomes and strategies. Here, we formulate feasible entanglement witnesses for bright squeezed vacuum in the form of the macroscopically populated polarization triplet Bell states. Their testing involves efficient photodetection and the measurement of the Stokes operators’ variances. We also calculate the measures of entanglement for these states such as the Schmidt number and the logarithmic negativity. Our results show that the bright squeezed vacuum degree of polarization entanglement scales as the mean photon number squared. We analyze the applicability of an operational analog of the Schmidt number.

I. INTRODUCTION

Quantum entanglement is a fascinating phenomenon, especially if it is observed at the macroscopic scale [1,2]. It emerges from the quantum superposition principle lying at the heart of quantum mechanics. If a two- or multimode superposition is highly populated, macroscopically entangled subsystems are created. A bipartite system maximally entangled in a given degree of freedom has two subsystems, for which the values of the degree of freedom are completely random but still perfectly correlated. Importantly, macroscopic quantum correlations can be revealed only by accurate measurement outcomes and strategies [3].

Entanglement is the basic resource for quantum-information processing, quantum communication, and other quantum technologies [4,5]. For this reason, there is an important need for efficient and reliable entanglement verification and quantification in various physical systems. This is done by the measurement of entanglement witnesses or measures [2]. While an entanglement witness only tells whether a system is entangled or not, an entanglement measure allows one to quantify the amount of entanglement. Unfortunately, most of the entanglement measures are not operational; i.e., they are not directly measurable in experiment. An operational measure of entanglement has been proposed for continuous variables such as wave vector or frequency [6], but it is absent for discrete variables of large dimension, e.g., $10^{10}$, such as a photon number.

The first condition for entanglement (inseparability), the Peres-Horodecki criterion, was formulated in terms of density matrices [7] and therefore was not operational. Later, other conditions were formulated in terms of measurable quantities such as variances or uncertainties of [8–11]. A general method of finding inseparability criteria for bipartite continuous variable systems, deeply related to the Peres-Horodecki condition, was proposed in Ref. [12] and then amended in Refs. [13,14]. It contains a hierarchy of inequalities for the measurable moments of creation and annihilation operators and generalizes the previously obtained conditions [8–11].

Recently, macroscopic states of light have generated a lot of interest and posed important questions in the scientific community. A possibility of verifying macroscopic entanglement became intriguing and was widely discussed [15–17]. Among macroscopic photonic states, two kinds of entangled states have been generated and reported: the micro-macro polarization singlet state, where a macroscopic qubit is entangled with a single photon [18], and entangled bright squeezed vacuum (BSV) states, macroscopic analogs of two-photon polarization Bell states [19]. An entanglement test with the micro-macro singlet state was performed [18], but later refuted [20]. It was shown that in the case of macroscopic states, inefficient detection may falsely reveal entanglement in separable states [21] and that it is incapable of grasping their quantum character [22]. New criteria for the test were formulated [20,23], but not put in experiments. Entanglement in BSV states is known as twin-beam multiphoton entanglement [24] and manifests itself in perfect polarization correlations between two macroscopically populated beams. The macroscopic singlet polarization Bell state [19] is formally equivalent to the singlet state of a large spin, with the spin value being hugely uncertain. An entanglement witness in the form of spin inequality for BSV has been discussed [25–27]. Recently, the nonseparability witness derived in Ref. [26] was tested in experiment [28] for this state. The form in which the nonseparability witness was derived is inapplicable to the other three BSV states (macroscopic triplet polarization Bell states) experimentally obtained in Ref. [19]. Experimental realization of entanglement measures, e.g., the Schmidt number or other measures based on the eigenvalues of the density operator, for entangled macroscopic states of light remains an open issue.

In this paper, we formulate feasible nonseparability (entanglement) witnesses for the macroscopic triplet polarization Bell states in the spirit of the Duan criterion. Their testing involves efficient photodetection and the measurement of the...
Stokes operators’ variances. We also calculate measures of entanglement for these states such as the logarithmic negativity and the Schmidt number as well as the operational measure $R$ introduced in Ref. [6]. Finally, we examine the dependence of the measure $R$ and its relation to the Schmidt number on the experimental conditions.

This paper is organized as follows. We start with introducing the basic properties of the bright squeezed vacuum in Sec. II. Section III is devoted to the derivation of the operational entanglement witnesses for the macroscopic triplet polarization Bell states. Next, in Sec. IV we discuss various entanglement measures: the Schmidt number, the effective Schmidt number, and the negativity. A possibility of their experimental verification is addressed. Finally, we present the conclusions in Sec. V.

II. ENTANGLED SQUEEZED VACUUM

Entangled four-mode BSV states of light are prepared in experiment by employing two two-mode BSV states [29] obtained via high-gain unseeded parametric down-conversion (PDC). In Ref. [19], they were created by overlapping on a polarization beam splitter two orthogonally polarized beams of frequency-nondegenerate squeezed vacuum.

The generation process, depending on the experimental conditions, is described by one of the following Hamiltonians:

$$\mathcal{H}_{\Phi, \pm} = \frac{i}{\hbar} g (a_1^\dagger b_Y^\dagger \pm a_1 b_Y^\dagger) + \text{H.c.},$$

$$\mathcal{H}_{\Phi, \pm} = \frac{i}{\hbar} g (a_1^\dagger b_H^\dagger \pm a_1 b_H^\dagger) + \text{H.c.},$$

(1)

where $g$ is the coupling constant proportional to the pump field, the PDC crystal length, and the second-order nonlinearity of the crystal. The down-converted photons carry linear polarizations $H$ (horizontal) and $V$ (vertical) and are emitted in two frequency modes described by the creation operators $a^\dagger$ and $b^\dagger$. The resulting states can be considered as macroscopic (multiphoton) generalizations of the two-photon polarization Bell states,

$$|\Psi^{(\pm)}_{\text{mac}}\rangle = e^{i\delta_1 a_1^\dagger b_Y^\dagger \pm a_1 b_Y^\dagger} + \text{H.c.} |\text{vac}\rangle,$$

$$|\Phi^{(\pm)}_{\text{mac}}\rangle = e^{i\delta_1 a_1^\dagger b_H^\dagger \pm a_1 b_H^\dagger} + \text{H.c.} |\text{vac}\rangle,$$

(2)

where $\delta_1 = \int g dt$ is the parametric gain coefficient.

The above equation allows one to determine their Schmidt decomposition. For the singlet state $|\Psi^{(\pm)}_{\text{mac}}\rangle$, the decomposition is known [5,28]. For $|\Psi^{(\pm)}_{\text{mac}}\rangle$, it has a similar form, so that both can be written as

$$|\Psi^{(\pm)}_{\text{mac}}\rangle = \sum_{n,m=0}^{\infty} (\pm 1)^n \sqrt{\lambda_n \lambda_m} |n,m\rangle_a |m,n\rangle_b,$$

(3)

where $\lambda_n \equiv \tanh^2 \delta_1 / \cosh^2 \delta_1$ and $|n,m\rangle_a \equiv |n\rangle_a \otimes |m\rangle_a$, denotes a two-mode Fock state with $n$ photons polarized horizontally and $m$ photons polarized vertically in beam $a$ (similarly for beam $b$). It is possible to factorize Eq. (3) further into two independent Schmidt decompositions, one of them involving modes $a_H$ and $b_Y$ and the other one modes $a_V$ and $b_H$ [28],

$$|\Psi^{(\pm)}_{\text{mac}}\rangle = |\Psi^+_1\rangle \otimes |\Psi^\pm_2\rangle,$$

$$|\Psi^+_1\rangle = \sum_{n=0}^{\infty} \sqrt{\lambda_n} |n\rangle_a |n\rangle_b,$$

$$|\Psi^\pm_2\rangle = \sum_{m=0}^{\infty} (\pm 1)^m \sqrt{\lambda_m} |m\rangle_a |m\rangle_b.$$

The Schmidt decompositions for the other two triplet states, $|\Phi^{(\pm)}_{\text{mac}}\rangle$, can be easily written by recalling that they are obtained from $|\Psi^{(+)}_{\text{mac}}\rangle$ by rotating the polarization. As a result, they will have the same form as the one for $|\Psi^{(\pm)}_{\text{mac}}\rangle$, but will be expressed in different polarization bases. For $|\Phi^{(+)}_{\text{mac}}\rangle$ it will be the right and left $(R,L)$ circular polarization and for $|\Phi^{(-)}_{\text{mac}}\rangle$ it will be the $\pm 45^\circ$ linear polarization basis.

The key difference between the two- and the four-mode BSV concerns entanglement. The two-mode BSV involves only photon-number entanglement. Namely, the photon number in beam $a$ with a fixed polarization (e.g., $H$) is unknown, but it is always equal to the photon number in beam $b$ having the orthogonal polarization ($V$). This state is known to approximate the maximally entangled EPR state in the high gain limit [30]. In the coordinate representation in the limit $\Gamma \to \infty$ the electric field quadratures become completely uncertain and $\delta$ correlated. The four-mode BSV is a product of the two such states and, thus, it simply provides two copies of it. However, it also contains polarization entanglement between beams $a$ and $b$. This polarization entanglement is probed through measuring the photon-number correlations present in the two-mode BSV [see Eq. (2)]. It is called twin-beam diphoton entanglement [24] and is most easy to notice if, e.g., $|\Psi^{(+)}_{\text{mac}}\rangle$ is rewritten as a superposition,

$$|\Psi^{(+)}_{\text{mac}}\rangle = \frac{1}{\cosh^2 \delta_1} \sum_{n=0}^{\infty} \sqrt{n+1} \tanh n \Gamma |\psi^{-}_{n}\rangle,$$

$$|\psi^{-}_{n}\rangle = \frac{1}{\sqrt{n+1}} \sum_{m=0}^{n} (-1)^m |n-m,m\rangle_a |m,n-m\rangle_b,$$

(5)

where $|\psi^{-}_{n}\rangle$ is an analog of a singlet state of two spin-$\frac{1}{2}$ particles. $|\Psi^{-}\rangle$ is invariant with respect to joint rotations of the polarization bases of both modes. Polarization of each beam separately is undetermined, but the polarizations of beams $a$ and $b$ are anticorrelated. The situation is similar for the triplet states. This explains why these states can be considered as macroscopic (multiphoton) generalizations of the two-photon polarization Bell states.

III. ENTANGLEMENT WITNESSES

Entanglement witnesses are sufficient conditions for entanglement. Although sometimes being inconclusive, they are so far the only practical option for proving the entanglement of multidimensional systems. Conclusive (necessary and sufficient) conditions of entanglement are formulated only for special classes of quantum systems. For instance, the Peres-Horodecki criterion [7] provides a necessary and sufficient condition for the entanglement of two- and three-dimensional systems. Its continuous-variable counterpart, formulated by
Simon [9], is in the general case also only a sufficient condition, but becomes necessary for Gaussian states. In practice, continuous-variable entanglement is often witnessed using the Duan et al. criterion [8], containing variances of sum and difference quadratures for the subsystems of a bipartite system.

In experiment, the output states $|\Psi_{mac}^{(\pm)}\rangle$ and $|\Phi_{mac}^{(\pm)}\rangle$ are given by a compound beam, comprising two independent frequency modes. Therefore, their Stokes operators $S_i$ are given by the sum of partial [31] Stokes operators for modes $a$ and $b$,

$$ S_i = S_i^a + S_i^b, \quad (6) $$

with $i = 0, 1, 2,$ and $3; S_0^a = a _{j} ^{†}a_H + a _{j} ^{†}a_V; S_0^b = a _{j} ^{†}a_H - a _{j} ^{†}a_V; S_2^a = a _{j} a_V + a _{j} a_H; \text{ and } S_2^b = i(a _{j} a_V - a _{j} a_H)$, and similarly for mode $b$.

The condition involving variances of the Stokes operators,

$$ \text{Var}(S_1) + \text{Var}(S_2) + \text{Var}(S_3) \geq 2S_0, \quad (7) $$

holds true for any separable state of subsystems $a$ and $b$ [26,28]. This fact allows one to formulate an entanglement witness operator:

$$ W_2 = (S_0^a + S_0^b - (S_2^a + S_2^b)) + (S_2^a + S_2^b - (S_0^a + S_0^b))^2 + (S_3^a + S_3^b - (S_3^a + S_3^b))^2 - 2S_0. \quad (8) $$

As usual, a negative mean value of the witness indicates entanglement.

It is worth noting that the sign of $\langle W_2 \rangle$ is invariant to the number of spatial and temporal modes because the overall state is a product, $\prod_i |\Psi_k^{(\pm)}\rangle$, of states for different modes that are pairwise entangled (between modes $a_{j, k}, a_{H, k}, b_{j, k}, \text{ and } b_{H, k}$), so both the Stokes variances and the mean photon number of the whole beam contain additive contributions of separate modes. However, multimode separable states do not necessarily have this property. Therefore, the witness is valid for spatially and temporally multimode beams, under the assumption that separate modes are independent (the overall state is a product) [32,33]. This property enabled its experimental testing [28] for the macroscopic BSV singlet state $|\Psi^{(-)}_{mac}\rangle$. Indeed, the macroscopic polarization singlet Bell state has noise completely suppressed in all Stokes observables $S_{1, 2, 3}$ [19]; hence $\langle W_2 \rangle = -2\langle S_0 \rangle < 0$.

At the same time, the witness (8) will not be negative for the three triplet states $|\Psi^{(+)}_{mac}\rangle$ and $|\Phi^{(\pm)}_{mac}\rangle$, since they have noise suppressed only in one Stokes observable [33]. However, all states in Eq. (2) can be transformed into each other by local polarization transformations and, thus, they contain the same amount of entanglement. Based on $W_2$, we further derive the witnesses applicable to the triplet states.

The $|\Psi^{(+)}_{mac}\rangle$ state and the singlet state are linked by the local unitary rotation $|\Psi^{(+)}_{mac}\rangle = U_0 |\Psi^{(-)}_{mac}\rangle$, where $U_0 = e^{i\pi b_{0}b_{0}}$, $U_0^\dagger U_0 = 1_b$, and $U_{a(b)}$ is a unity operator acting on beam $a$ ($b$). This follows from the fact that the rotation $I_3 \otimes U_0$ transforms the Hamiltonian $H_{-\psi}$ into the Hamiltonian $H_{+\psi}$. In experiment, this transformation is easily realized by means of a half-wave plate inserted into beam $b$ with the optic axis oriented vertically or horizontally. Thus, the entanglement witness $W_{T1}$ for $|\Psi_{mac}^{(+)}\rangle$ equals $W_{T1} = U_0 W_2 U_0^\dagger$, which yields

$$ W_{T1} = (S_0^b + S_0^b - (S_2^b + S_2^b))^2 + (S_2^b + S_2^b - (S_0^b + S_0^b))^2 + (S_3^b + S_3^b - (S_3^b + S_3^b))^2 - 2S_0. \quad (9) $$

Of course, in theory the measurement of the witness $W_{T1}$ for the state $|\Psi^{(+)}_{mac}\rangle$ is equivalent to the measurement of the witness $W_2$ for the singlet state, $\langle \Psi^{(-)}_{mac} | W_2 | \Psi^{(-)}_{mac} \rangle = \langle \Psi^{(+)}_{mac} | W_{T1} | \Psi^{(+)}_{mac} \rangle$.

Entanglement witnesses for the other two macroscopic polarization triplet Bell states, $|\Phi^{(\pm)}_{mac}\rangle$, can be easily obtained by recalling that the triplet states are transformed into each other via global rotations in the Stokes space. In particular, $|\Phi^{(\pm)}_{mac}\rangle$ is obtained from $|\Psi^{(+)}_{mac}\rangle$ by a $\pi/2$ rotation around the $S_2$ axis [33]. This rotation can be realized by a quarter-wave plate with the optic axis at angle $45^\circ$ inserted into both beams $a$ and $b$. The resulting witness can be obtained from Eq. (9) by changing variables $S_1 \rightarrow S_3$ and $S_3 \rightarrow -S_1$:

$$ W_{R2} = (S_0^b - S_0^b - (S_2^b - S_2^b))^2 + (S_2^b - S_2^b - (S_0^b - S_0^b))^2 + (S_3^b - S_3^b - (S_2^b + S_2^b))^2 - 2S_0. \quad (10) $$

Finally, $|\Phi_{mac}^{(-)}\rangle$ is obtained from $|\Psi_{mac}^{(+)}\rangle$ by a $\pi/2$ rotation around the $S_1$ axis [33]. It can be realized by a $\pi/2$ rotator (a quartz crystal or a Faraday cell). The witness has the form

$$ W_{T3} = (S_0^b - S_0^b - (S_2^b - S_2^b))^2 + (S_2^b + S_2^b - (S_0^b + S_0^b))^2 + (S_3^b - S_3^b - (S_2^b + S_2^b))^2 - 2S_0. \quad (11) $$

Indeed, the witnesses (9)–(11) have negative mean values for the corresponding triplet states, because $\text{Var}(S_1^b + S_1^b) = \text{Var}(S_2^b - S_2^b) = \text{Var}(S_3^b - S_3^b) = 0$ for state $|\Phi_{mac}^{(+)}\rangle$, $\text{Var}(S_1^b - S_1^b) = \text{Var}(S_2^b - S_2^b) = \text{Var}(S_3^b + S_3^b) = 0$ for state $|\Phi_{mac}^{(-)}\rangle$, and $\text{Var}(S_3^b - S_3^b) = \text{Var}(S_2^b + S_2^b) = \text{Var}(S_3^b - S_3^b) = 0$ for state $|\Phi_{mac}^{(\pm)}\rangle$.

It is worth mentioning that the conditions for entanglement given by witnesses (8)–(11) look very similar to the Duan condition as they contain the variances of sum and difference operators for subsystems $A$ and $B$. At the same time, the relation between the Stokes operators from Eqs. (8)–(11) and the quadrature operators from the Duan condition is only known for the case of light with a bright polarized coherent component and not for our case.

In Fig. 1 we depict an experimental setup where the witnesses $W_2$ can be tested. It enables simultaneous measurement of various Stokes observables for beams $a$ and $b$. The complete test consists of the measurement of the variances for combinations of partial Stokes operators, $\text{Var}(S_{0, 1, 2, 3}^a \pm S_{0, 1, 2, 3}^b)$, $i = 1, 2,$ and 3, and the total photon number $\langle S_0 \rangle$.

The measurement should consist of three series. In each series, one of the variances entering Eqs. (9)–(11) is measured. This implies certain positions of the wave plates in the Stokes measurement setup. For the $S_1$ measurement, both wave plates should have their optic axes horizontal. For the $S_2$ measurement, the HWP should have the optic axis at $22.5^\circ$ and the QWP should have the optic axis at $45^\circ$. Finally, for...
tanglement witnesses, Eqs. (9)–(11). In each beam, there is a Stokes measurement setup: DM, dichroic mirror; PBS, polarizing beam splitter; HWP, half-wave plate; QWP, quarter-wave plate; D, detectors. The signals from the two detectors in each arm are subtracted to obtain the Stokes observables.

the $S_3$ measurement, the HWP should have the optic axis horizontal and the QWP should have the optic axis at 45°.

The variances should be calculated by averaging over a large number of pulses.

IV. ENTANGLEMENT MEASURES

A. Schmidt number

One of the well-known entanglement measures is the number of nonzero Schmidt coefficients $\sqrt{\lambda_i}$ in the Schmidt decomposition [2]. It is called the Schmidt rank or number. In the case of a maximally entangled bipartite system with symmetrical subsystems, defined in the Hilbert space $\mathcal{H} = \mathcal{C}^d \otimes \mathcal{C}^d$, all Schmidt coefficients have to be equal and the Schmidt number is $K = d$, where $d$ is the dimensionality of the subsystem. In general, $1 \leq K \leq d$ and a state is separable if $K = 1$. To quantify entanglement in systems with infinite-dimensional Hilbert space such as BSV, for which $K = \infty$, another measure is more appropriate, which we further call the effective Schmidt number. It is defined as follows [34,35]:

$$R \equiv 1/\text{Tr}(\rho^2) = 1/\sum \lambda_i \lambda_i,$$

(12)

where $\rho$ is a density operator and $\sum \lambda_i = 1$. This definition coincides with the original definition of $K$ in the following way. For a separable state $K = 1$. For a maximally entangled state with all Schmidt coefficients equal and $d \to \infty$ (such a state does not exist because it is not normalizable) we would obtain $K = d \to \infty$. Otherwise, $K$ is finite even if the number of Schmidt coefficients $\sqrt{\lambda_i}$ is infinite.

The effective Schmidt number for the $|\Psi^{(\text{mac})}\rangle$ state is the product of the effective Schmidt numbers for subsystems $|\Psi^{(1,2)}\rangle$, $K_1 = K_2 = (\sum_n \lambda_n^2)^{-1}$, which yields $K = (1 + 2 \sinh^2 \Gamma)^2$. Since all the macroscopic polarization Bell states (2) have the same form of the Schmidt decomposition, the effective Schmidt number for all of them is the same, $\hat{K} = (1 + 2N_0)^2$, where $N_0 = \sinh^2 \Gamma$ is the photon population in each mode $a_H$, $a_V$, $b_H$, and $b_V$. For bright states $N_0 \gg 1$ and $\hat{K} \approx 4N_0^2$, hence the degree of polarization entanglement grows quadratically with the mean photon number.

B. Negativity

Other widely used and easily calculated measures of entanglement are the negativity and the logarithmic negativity which gives an upper bound for distillable entanglement [36]. For a bipartite quantum state $\rho$, they are defined as $N(\rho) = \parallel \rho^{PT} \parallel _1 - 1$ and $E_N(\rho) = \log_2 \parallel \rho^{PT} \parallel _1$, where PT denotes partial transposition with respect to one of the subsystems and $\parallel A \parallel _1 = \text{Tr} \sqrt{A^\dagger A}$ the trace norm of the operator $A$.

Particularly useful is the fact that the trace norm is factorizable, $\parallel \rho_1 \otimes \rho_2 \parallel _1 = \parallel \rho_1 \parallel _1 \parallel \rho_2 \parallel _1$, and that the logarithmic negativity is additive, i.e., $E_N(\rho_1 \otimes \rho_2) = E_N(\rho_1) + E_N(\rho_2)$ [36].

Since a four-mode BSV is the product of two entangled bipartite subsystems as in Eq. (4), it is sufficient to determine the negativities of each of them, $\rho_1$ and $\rho_2$, separately. Performing partial transpositions with respect to the subsystems $b_V$ and $b_H$, respectively, we obtain $\parallel \rho_1^{PT} \parallel _1 = \parallel \rho_2^{PT} \parallel _1 = \left(\sum_{n=0}^{\infty} \sqrt{K_n} \lambda_n \right)^2 = e^{2\Gamma}$. Thus, for the four-mode BSV, the negativity equals $N(\rho) = e^{2\Gamma} - 1$. For high gain, $N(\rho) \simeq 16 \sinh^2 \Gamma = 16 N_0^2$. Again, the quadratic dependence of the degree of entanglement on the population is observed. We notice that $N(\rho) = 4\hat{K}$. The logarithmic negativity takes the value $E_N(\rho) = 4\Gamma/\ln 2$. In comparison, for two-mode BSV it equals $2\Gamma/\ln 2$ [37], which shows that BSV macroscopic Bell states contain (with respect to the logarithmic negativity) twice more entanglement than the usual two-mode BSV. This is understandable: since the four-mode BSV consists of two copies of entangled states, twice more entanglement can be distilled from it than from a single copy.

C. Fedorov ratio

Both the Schmidt number and the negativity are not operational quantifiers of entanglement as they cannot be directly measured in experiment. For a bipartite system entangled in a continuous variable, an operational measure has been proposed [6], called the Fedorov ratio. It is defined in the spirit of the entropy of entanglement with the advantage of being directly measurable in experiment. Further, we try to adapt this measure for characterizing BSV entangled in the photon number, a variable that is discrete but can be viewed as pseudocontinuous when large numbers are involved.

Consider a pure bipartite quantum system, entangled in a continuous variable $v$, with the state vector

$$|\Psi\rangle = \int d\nu_a d\nu_b F(\nu_a, \nu_b) |\nu_a\rangle |\nu_b\rangle.$$

(13)

The variable $\nu_a$ can be characterized by its marginal probability distribution $P(\nu_a) = \int d\nu_b F(\nu_a, \nu_b)^2$, with the standard deviation $\Delta \nu_a$ and the conditional probability distribution $P(\nu_a | \nu_b) = |F(\nu_a, \nu_b)|^2$, evaluated for a certain value of $\nu_b$, with the width $\delta \nu_b$.

The effective Schmidt number $\hat{K}$ for the state $|\Psi\rangle$ is very well approximated by the ratio $R$ [6] defined as

$$R_{\nu} = \frac{\Delta \nu_a}{\delta \nu_b}.$$

(14)

Of course, equivalently the variable $\nu_b$ may be involved in this definition, instead of $\nu_a$. The parameter $R$, known as the Fedorov ratio, can be easily obtained in experiment and is hence an operational measure of entanglement. Note however,
that this measure is only defined for pure states, so to be operational stricto sensu it has to be supplemented with an experimental proof of the purity of the global state. The possibility to generalize the Fedorov ratio to mixed states is an open question. For instance, it has been measured for the cases of wave vector [38] and frequency entanglement [39]. For Gaussian states, the Fedorov ratio exactly coincides with the effective Schmidt number [40].

We adopt definition (14) for the photon-number variable and its probability distributions in the following way:

$$R_n = \frac{\Delta n_a}{\delta n_a},$$

(15)

where $\Delta n_a$ is the width of the marginal photon-number distribution in beam $a$, while $\delta n_a$ is the width of the photon-number distribution in beam $a$ under the condition that a certain photon number $n_a$ has been measured in beam $b$.

Similarly to the effective Schmidt number, the ratio $R_n$ for each of the macroscopic Bell states is a product of the ratios $R_{01}$ and $R_{02}$ of its two subsystems. For instance, for the state $|\Psi_{mac}^{(1)}\rangle$, the states $|\Psi_{mac}^{(2)}\rangle$ in Eq. (4) have the same Fedorov ratios. They can be easily calculated by noticing that the marginal distribution $P(n_a)$ is a geometric one,

$$P(n_a) = \frac{(\tanh \Gamma)^{n_a}}{\cosh^2 \Gamma},$$

(16)

while the conditional distribution is given by the Kronecker $\delta$,  

$$P(n_a | n_b) = \delta_{n_a,n_b},$$

(17)

These distributions are schematically shown in Fig. 2.

Assuming that the width of the discrete $P(n_a | n_b)$ distribution is unity, the ratios $R_{01}$ and $R_{02}$ are given by the standard deviation of the geometric distribution (16), $R_{01,2} = \sqrt{2} \sinh^2 \Gamma$. Finally, the Fedorov ratio for $|\Psi_{mac}^{(1)}\rangle$ equals

$$R_n = 2N_0^2.$$

(18)

Thus, in the case of high population, the operational measure $R_n$ differs from the effective Schmidt number $\bar{K}$ by only a constant factor $1/2$.

The $R_n$ ratio for entangled BSV states can be measured with the help of the setup shown in Fig. 1. The orientation of the HWP and the QWP should be such that proper polarization bases are chosen in the arms $a$ and $b$. For instance, in the case of the singlet state $|\Psi_{mac}^{(-1)}\rangle$, the plates can be oriented in any way but similarly for arms $a$ and $b$. The ratios $R_{01}$ and $R_{02}$ can then be measured independently using pairs of detectors $D_{a1}, D_{b1}$ and $D_{a2}, D_{b2}$. For each pair, after acquiring a certain (large) number of pulses, the photon-number distributions should be analyzed and the conditional and unconditional widths should be measured. In practice, because the detectors do not distinguish between close photon numbers, the photon-number distribution should be binned in intervals of about 200 photons [28].

All the above-considered measures of entanglement for the case of $|\Psi_{mac}^{(1)}\rangle$ states are plotted in Fig. 3 as functions of the mean photon number.

**D. Effective dimensionality of BSV Hilbert space**

Entangled BSV states of light are considered to be macroscopic generalizations of polarization singlet or triplet Bell states due to symmetry reasons. Their polarization correlations are probed through photon-number measurements in polarization modes. Since the Hilbert space of these states is complex [41] and infinite, it is interesting to understand and quantify the amount of their accessible entanglement, for example, by comparison with finite-dimensional systems, where the notion of a maximally entangled state is well understood. The degree of polarization entanglement in the logarithmic scale for the four-mode entangled BSV state as a function of the average photon number $N_0$, the negativity $N$ (solid line), the effective Schmidt number $\bar{K}$ (dashed line), and the Fedorov ratio $R_n$ (dotted line).

All the above-considered measures of entanglement for the case of $|\Psi_{mac}^{(1)}\rangle$ states are plotted in Fig. 3 as functions of the mean photon number.

**FIG. 3.** The degree of polarization entanglement in the logarithmic scale for the four-mode entangled BSV state as a function of the average photon number $N_0$, the negativity $N$ (solid line), the effective Schmidt number $\bar{K}$ (dashed line), and the Fedorov ratio $R_n$ (dotted line).

Entangled BSV states of light are considered to be macroscopic generalizations of polarization singlet or triplet Bell states due to symmetry reasons. Their polarization correlations are probed through photon-number measurements in polarization modes. Since the Hilbert space of these states is complex [41] and infinite, it is interesting to understand and quantify the amount of their accessible entanglement, for example, by comparison with finite-dimensional systems, where the notion of a maximally entangled state is well understood.

We propose a rough estimate for the dimensionality $d$ of the Hilbert space of $|\Psi_{mac}^{[1]}\rangle$, depending on the gain $\Gamma$, based on the following argument. In two-photon experiments, a small gain $\Gamma \lesssim 10^{-3}$ is used to produce a superposition of the vacuum (the dominant component) and a biphoton. All higher-order contributions are largely suppressed. Truncation of the Fock states of order higher than one in Eq. (3), for a given gain, is justified if the normalization of the truncated state is preserved to a good approximation, given by the parameter $\epsilon$,

$$\lambda_0^2 + 2\lambda_0\lambda_1 = 1 - \epsilon.$$

(19)

For the two-photon case $\epsilon \approx \Gamma^4$. However, it is known that in these experiments four- and six-photon components are observed as well, which is manifested as, e.g., a decrease in the interference visibility for relatively bright PDC sources [42]. Thus, in our example, the value of $\epsilon$ shows, for a given gain, how well the outcome state from the PDC crystal can be approximated by a superposition of a two-photon state and the vacuum. Of course, the smaller $\epsilon$ is, the better the approximation gets.
By analogy with the low-gain case, for any value of $\Gamma$ we can locally restrict the state (3) to some finite-dimensional Hilbert space $H = C^{d_a} \otimes C^{d_b}$, where $d_a$ and $d_b$ denote dimensionals of beams $a$ and $b$. Then, $1 - \epsilon$ gives the probability to find the state in $H$. The natural choice for the subspace $C^{d_a}$ is to keep in $|\Psi_{\text{mac}}^{(\pm)}\rangle$ only those components which have a limited number of photons in beam $a$, $a_{\mu} a_{\mu}^\dagger + a_{\nu} a_{\nu}^\dagger \leq N_{\text{max}}$, and similarly for $C^{d_b}$ and beam $b$. This restriction implies the following normalization condition for the truncated state $|\Psi_{\text{mac}}^{(\pm)}\rangle$:

$$
\sum_{n=0}^{N_{\text{max}}} \lambda_n \sum_{m=0}^{N_{\text{max}}-n} \lambda_m = 1 - \epsilon. \quad (20)
$$

The normalization is calculated over the sectors of the density matrix with a fixed photon number, so that $n + m \leq N_{\text{max}}$. This enables one to determine the dimensionality of $C_{d_a}^{d_a}$, $d = d_a = d_b = (N_{\text{max}} + 1)(N_{\text{max}} + 2)$, and dependence of $N_{\text{max}}$ on the average population $N_0$. Using $\text{tanh}^2 \Gamma = N_0/(N_0 + 1)$ we turn Eq. (20) into

$$
\epsilon = \left( \frac{N_0}{N_0 + 1} \right)^{1 + N_{\text{max}}} \left[ N_{\text{max}} + 2 - \frac{N_0}{N_0 + 1}(N_{\text{max}} + 1) \right] \quad (21)
$$

and obtain a linear relation between $N_{\text{max}}$ and $N_0$. This allows one to express the dimensionality for a large population as $d \approx \frac{e^{-\alpha}}{\epsilon} N_0^2$, where $\alpha$ is a function of $\epsilon$ given by the equation $\epsilon = e^{-\alpha} (\alpha + 1)$. For example, if $\epsilon = 10^{-12}$ as in the two-photon case, $\alpha \approx 31$. If $\epsilon = 10^{-12}$, we obtain $\alpha \approx 7.4$. Please note the quadratic scaling of the dimensionality with $N_0$.

The above estimations of the effective dimensionality of the Hilbert space for BSV are useful for reconstructing its density matrix, or the most significant part of it. It would provide almost full information about the joint photon-number distribution and could enable calculation of the entanglement measures based on the eigenvalues of the density operator obtained from the experimental data.

Now we investigate the amount of entanglement in the truncated state:

$$
|\Psi_{\text{mac}}^{(\pm)}\rangle = \frac{1}{\sqrt{1 - \epsilon}} \sum_{n,m=0}^{n+m \leq N_{\text{max}}} (\pm 1)^n \sqrt{\lambda_n \lambda_m} |n,m\rangle_a |m,n\rangle_b. \quad (22)
$$

Since the restriction is local, the amount of entanglement in $|\Psi_{\text{mac}}^{(\pm)}\rangle$ gives a lower bound on the overall entanglement in $|\Psi_{\text{mac}}^{(\pm)}\rangle$. The effective Schmidt number for the truncated state $K_T = (1 - \epsilon)^2 \sum_{n,m=0}^{n+m \leq N_{\text{max}}} \lambda_n \lambda_m$ fulfills $\left( \frac{1}{\epsilon} - 1 \right) K_T < K_T < (1 - \epsilon) K_T$. We look at how far is the restricted state $|\Psi_{\text{mac}}^{(\pm)}\rangle$ from a maximally entangled state in $H$ as a function of $\epsilon$, where $H = C^d \otimes C^d$. This question may be understood in a more practical way as showing how good our approach is for creating a maximally entangled state of dimension $d$. Since the maximally entangled states are defined with respect to the dimensionality of their Hilbert space, we depict $K_T/d$ as a function of $\epsilon$ in Fig. 4. We notice that at high gain, a perfect maximally entangled state $(K_T/d = 1)$ is obtained only with $\epsilon$ approaching 1. This is because $|\Psi_{\text{mac}}^{(\pm)}\rangle$ is a superposition of singlet states of different spin values and thus belongs to Hilbert spaces with different dimensionalities.

![Figure 4](image-url)
ACKNOWLEDGMENTS

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[32] This assumption is indeed necessary: Consider the following multimode state with two photons on each side:

$$\frac{1}{2}(a^+_{H,1}a^+_{V,2} - a^+_{V,1}a^+_{H,2})(0)(b^+_{H,1}b^+_{V,2} - b^+_{V,1}b^+_{H,2})(0).$$

For this state the witness (8) takes a negative value ($W_2 = -8$, while the subsystems $A$ and $B$ are separable. In the multimode case the Stokes operators in Eq. (6) are given by the sum of Stokes operators for all different subsystems ($a_k$ and $b_k$), and the violation of the witness only ensures that there is entanglement between some subsystems.

[41] We mean that the states are defined in the photon-number spaces $a_H$, $a_V$, $b_H$, and $b_V$.